DEPARTAMENTO DE MATEMÁTICA
DOCUMENTO DE TRABAJO

"A Hahn-Banach Theorem for Integral Polynomials"

Daniel Carando e Ignacio Zalduendo

D.T.: N° 10 Noviembre 1997

Vito Dumas 284, (B1644BID) Victoria, Pcia. de Buenos Aires, Argentina
Tel: 4725-7053 Fax: 4725-7010
Email: matematica@udesa.edu.ar

A Hahn-Banach Theorem for Integral Polynomials

Daniel Carando and Ignacio Zalduendo

Abstract

We study the problem of extendibility of polynomials over Banach spaces: when can a polynomial defined over a Banach space be extended to a polynomial over any larger Banach space? To this end, we identify all spaces of polynomials as the topological duals of a space S spanned by evaluations, with Hausdorff locally convex topologies. We prove that all integral polynomials over a Banach space are extendible. Finally, we study the Aron-Berner extension of integral polynomials, and give an equivalence for non-containment of ℓ_1.

[1]

Universidad de SanAndrés

[1]1991 Mathematics Subject Classification: Primary 46G20, Secondary 46B
Key words and phrases: extension of polynomials, containment of ℓ_1.

Introduction

A natural question concerning scalar-valued continuous homogeneous polynomials over a Banach space E, is whether they can be extended to a larger space G, much as linear forms can be extended by using the Hahn-Banach theorem. It is well-known that the answer to this question is in general, no. There are several positive answers for particular situations. Most of these rely –explicitly or not– on the existence of a linear extension morphism for linear functionals $E' \longrightarrow G'$. This is the case of $G = E''$ ([3], [9]), of more general but similar constructions ([15], [27]), and even of the ultrapower methods employed by [13] and by [20], which can be seen to be related to the existence of such an extension morphism. This is of course stronger than the Hahn-Banach theorem, and is in fact equivalent to E'' being complemented in G'' (see also [19] and [21]). Moreover, it is equivalent to the existence of a linear extension morphism for continuous homogeneous polynomials $P(^kE) \longrightarrow P(^kG)$. Here we are concerned with true Hahn-Banach type extensions; we want sufficient conditions for the extension of each P, not a linear morphism that will extend all of them.

Our approach consists in identifying polynomials with linear functionals and using the Hahn-Banach theorem to extend these. Thus in §1 we study preduals of several different spaces of polynomials. Note that we are only interested in identifying polynomials with linear forms, so our identifications of spaces of polynomials with spaces of linear forms need only be algebraic. It is important however that all polynomials be identified with linear forms over the same space, and that the different classes of polynomials appear as we vary the topology over this space.

In a recent paper, Kirwan and Ryan [18] study the space of all "extendible" polynomials (i.e. those which can be extended to any larger space). We prove in §2 that all integral polynomials are extendible. The converse is false; we show that there are extendible polynomials over c_0 which are not integral. The paper ends with a discussion of the Aron-Berner extension of integral polynomials in §3.

We refer to [12] and [23] for notation and results regarding polynomials in general, and to [1] and [2] for more on integral polynomials.

The authors have a marked tendency to ask their friends as soon as they find something they don't understand. Thus they have accumulated a large debt in answers to e-mail consultations. It is a pleasure to thank Richard Aron, Ray Ryan, Seán Dineen, José M. Ansemil, Jesús Angel Jaramillo and Klaus Floret for many answers.

The idea of using the Michael selection theorem was suggested by Richard Aron to the second author one sunny winter afternoon in the Buenos Aires botanical gardens, and substantial improvements in §3 are due to Klaus Floret.

Universidad de SanAndrés

1 Preduals of spaces of polynomials

It is well known [25] that the space $P(^kX)$ of k-homogeneous polynomials over a Banach space X is the dual space of the space of symmetric k-tensors of elements of X with the projective topology. $P(^kX)$ has also been viewed ([24], [17]) as the dual of a space spanned by evaluations. We take up the latter presentation here, for this will enable us to give a predual of the space of integral polynomials, an essential tool in our extension process of such polynomials.

Given an element $x \in X$, 'evaluation at x' is a continuous linear form over the space of k-homogeneous polynomials. We denote this evaluation morphism by e_x. The norm of e_x as an element of $P(^kX)'$ is $\|x\|^k$. Denote by S the linear subspace of $P(^kX)'$ spanned by all evaluations at points of X. This is a (non-closed) subspace whose elements s have non-unique representations of the form

$$s = \sum_{j=1}^{n} e_{x_j},$$

(for complex X) and the norm of such an element is

$$\|s\| = \sup\{|\sum_{j=1}^{n} P(x_j)| : P \in P(^kX) \text{ of norm one }\},$$

which is independent of the representation of s. Note that when X is a real Banach space and k is even, the representation must take into account the signs, so $s = \sum_{j=1}^{n} e_{x_j} - \sum_{i=1}^{m} e_{y_i}$. Our results are valid for both the real and complex case, but we will use only the complex-case notation. Note also that for any scalar λ, $e_{\lambda x} = \lambda^k e_x$. The dual of S can be readily seen to be $P(^kX)$.

We will need to consider other topologies on the vector space S. Recall [14] that the Borel transform $B : P(^kX)' \longrightarrow P(^kX')$ is the linear map given by $B(T)(\gamma) = T(\gamma^k)$. A linear form T is in the kernel of the B if and only if T is zero over the space of approximable polynomials. Since generally not all polynomials are approximable, B is rarely one-to-one. However, the following lemma assures that when restricted to the space S, the Borel transform is always one-to-one.

Lemma [17]: Let $x_1, \ldots, x_n \in X$. If $\sum_{j=1}^{n} \gamma(x_j)^k = 0$ for all $\gamma \in X'$, then $\sum_{j=1}^{n} P(x_j) = 0$ for all $P \in P(^kX)$.

This is a purely algebraic lemma and does not require that the polynomials be approximable in any way by the γ's, or in fact, even continuous. For an element $s \in S$, it is equivalent to 'be zero' as a function applied to polynomials or as a function applied to linear functionals. In view of the injectivity of the Borel transform over S, we may now consider the same set S but with the topology induced by it as a subspace of $P(^kX')$. Each evaluation e_x has the same norm as before, but the situation is totally different for sums of such evaluations, the new norm being

$$\|s\| = \sup\{|\sum_{j=1}^{n} \gamma(x_j)^k| : \gamma \in X' \text{ of norm one }\}.$$

Note that S with this norm can be seen to be isomorphic to the symmetric ε-tensor product of X. The dual of S with this norm is thus the space of integral polynomials over X.

Proposition: S' is isometrically isomorphic to $P_I(^kX)$.

Proof: Denote with $B_{X'}$ the closed unit ball of X' considered with the w^*-topology. Then S is a subspace of $C(B_{X'})$. If $T \in S'$, we may extend T by Hahn-Banach to $\overline{T} \in C(B_{X'})'$, which by the Riesz representation theorem may be identified with a complex regular Borel measure μ on $B_{X'}$. We define then a k-homogeneous integral polynomial over X, P_T, by

$$P_T(x) = \int_{B_{X'}} \gamma(x)^k d\mu(\gamma).$$

Note that for all $x_1, \ldots, x_n \in X$, if $s = \sum_{j=1}^{n} e_{x_j} \in S$,

$$T(s) = \int_{B_{X'}} s(\gamma) d\mu(\gamma) = \int_{B_{X'}} \sum_{j=1}^{n} \gamma(x_j)^k d\mu(\gamma) = \sum_{j=1}^{n} \int_{B_{X'}} \gamma(x_j)^k d\mu(\gamma) = \sum_{j=1}^{n} P_T(x_j).$$

In particular, $P_T(x) = T(e_x)$, so P_T does not depend on any particular extension of T. Also, the mapping $T \longmapsto P_T$ from S' to $P_I(^kX)$ is linear, onto, and

$$\|P_T\|_I = \inf\{\|\nu\| : \nu \text{ represents } P_T\} \leq \|\mu\| = \|\overline{T}\| = \|T\|.$$

On the other hand, for any ν representing P_T, we have

$$\|T\| = \sup_{\|s\| \leq 1} |T(s)| = \sup_{\|s\| \leq 1} \left| \int_{B_{X'}} s(\gamma) d\nu \right| \leq \sup_{\|s\| \leq 1} \int_{B_{X'}} |s(\gamma)| d|\nu| \leq \int_{B_{X'}} d|\nu| = \|\nu\|,$$

so that $\|T\| \leq \inf\{\|\nu\| : \nu \text{ represents } P_T\} = \|P_T\|_I$.

Note that an immediate consequence of the proposition is that the image of the Borel map $B : P(^kX)' \longrightarrow P(^kX')$ is the space of integral k-homogeneous polynomials over X'.

Although we are mainly concerned with integral polynomials, we will take a moment here to comment on S as the predual of other spaces of polynomials. Note first that the mapping $T \mapsto P_T$ where $P_T(x) = T(e_x)$ (as in the theorem), is an isomorphism between the algebraic dual of S and the space of all (not necesarily continuous) k-homogeneous polynomials over X (we will denote its inverse by $P \mapsto T_P$). Imposing more or less stringent continuity conditions on T will produce different kinds of polynomials. We prove next that all spaces of polynomials are produced in this way.

Proposition: If Z is any subspace of $P(^kX)$ containing the finite type polynomials, Z is (algebraically) isomorphic to $(S, \tau)'$, where τ is a Hausdorff locally convex topology on S.

Proof: Define for each $Q \in Z$, the seminorm

$$p_Q(s) = \left| \sum_{j=1}^{n} Q(x_j) \right|,$$

where $s = \sum_{j=1}^{n} e_{x_j}$. Let τ be the locally convex topology generated by all such seminorms with $Q \in Z$. τ is Hausdorff because of the Lemma, bearing in mind that Z contains γ^k for all $\gamma \in X'$. If T_P is τ-continuous, there are $Q_1, \ldots, Q_m \in Z$ and a $c > 0$ such that for all $s \in S$,

$$|T_P(s)| \leq c \max\{p_{Q_1}(s), \ldots, p_{Q_m}(s)\}$$

Thus $\cap_{i=1}^{m} Ker T_{Q_i} \subset Ker T_P$, and therefore there are scalars $a_1, \ldots, a_m$ such that $T_P = \sum_{i=1}^{m} a_i T_{Q_i}$ and $P = \sum_{i=1}^{m} a_i Q_i \in Z$.

Of course the seminorms in the above proposition are not really very illuminating, so we want to consider some more concrete topologies on S. For instance, since S may be viewed as a subspace of the space of holomorphic functions on X', we may consider on S the topology of pointwise convergence τ_p, and the usual topologies of infinite dimensional holomorphy: the compact-open topology τ_0, the Nachbin topology τ_ω, and the τ_δ topology. For these last two, note that the norm topology on $P(^kX')$ coincides with τ_ω, and that $\tau_\omega = \tau_\delta$ on spaces of polynomials [12], so we have $(S, \tau_\omega)' = (S, \tau_\delta)' = P_I(^kX)$. For τ_p and τ_0 we have the following proposition.

Proposition: The mapping $T \mapsto P_T$ produces the following (algebraic) identifications.
i) $(S, \tau_p)' = P_f(^kX)$.
ii) $(S, \tau_0)' \subset P_\omega(^kX)$, in general strictly.

Proof: i) If $P = \gamma^k$ with $\gamma \in X'$, T_P is clearly continuous in the topology of pointwise convergence on $B_{X'}$, so for any polynomial Q in the space $P_f(^kX)$ which is spanned by such functions, T_Q is continuous in this topology. For the other inclusion, if T_Q is continuous in the topology of pointwise convergence, there is a $c > 0$ and $\gamma_1, \ldots, \gamma_m \in B_{X'}$ such that for all $s \in S$,

$$|T_P(s)| \leq c \max\{|s(\gamma_1)|, \ldots, |s(\gamma_m)|\}.$$

As above, $\cap_{i=1}^{m} Ker T_{\gamma_i^k} \subset Ker T_P$, so $P = \sum_{i=1}^{m} a_i \gamma_i^k$.

ii) If T is τ_0-continuous, then for some compact $K \subset X'$ and all $s \in S$, $|T(s)| \leq \sup_K |s(\gamma)|$. In particular, for $x \in X$,

$$P_T(x) \leq \sup_K |\gamma(x)|^k$$

so P_T is K-continuous in the sense of [7], and therefore weakly continuous on bounded subsets of X'. Note that since the τ_ω topology is stronger than the τ_0 topology, any $T \in (S,\tau_0)'$ corresponds to an integral polynomial. But not all weakly continuous polynomials are integral, so the inclusion is generally strict. Indeed, $(S,\tau_0)'$ corresponds to integral polynomials represented by regular Borel measures over compact subsets of X'.

We end this section by using the results of ([7], [5], and [26]) to present $P_\omega({}^kX)$ as the dual of (S,t) where t is a locally convex topology defined by more manageable seminorms than those given above for an arbitrary Z.

In [7] the following notion of K-continuous polynomial is introduced: given a compact subset K of X', a k-homogeneous polynomial P is said to be K-continuous if there is a constant $c>0$ such that for all $x \in X$

$$|P(x)| \leq c\|x\|_K^k,$$

where $\|x\|_K = \sup_{\gamma\in K}|\gamma(x)|$. They have shown that a k-homogeneous polynomial P is in $P_\omega({}^kX)$ if and only if it is K-continuous for some compact K. Now, given any compact $K \subset X'$ and $s \in S$, set

$$p_K(s) = \inf\{\sum_{j=1}^n \|x_j\|_K^k\}$$

where the infimum is taken over all possible representations of $s = \sum_{j=1}^n e_{x_j}$. We now define t on S as the locally convex topology generated by the seminorms p_K, $(K \subset X')$. Then

Proposition: $(S,t)'$ is (algebraically) $P_\omega({}^kX)$.

Proof: P is K-continuous if and only if T_P is p_K-continuous: if $|P(x)| \leq c\|x\|_K^k$ for all $x \in X$, then for any $s \in S$ and any representation of $s = \sum_{j=1}^n e_{x_j}$,

$$|T_P(s)| = \left|\sum_{j=1}^n P(x_j)\right| \leq \sum_{j=1}^n |P(x_j)| \leq c\sum_{j=1}^n \|x_j\|_K^k$$

thus $|T_P(s)| \leq cp_K(s)$. On the other hand, if T_P is p_K-continuous then in particular we have, for any $x \in X$, $|P(x)| = |T_P(e_x)| \leq cp_K(e_x) \leq c\|x\|_K^k$.

2 Extension of polynomials

In this section, we consider the problem of extending a polynomial defined on a Banach space E to a larger Banach space G. Thus, we will use the "linearization" of different types of polynomials presented in §1 in conjunction with the Hahn-Banach extension theorem for locally convex spaces, to produce extensions of polynomials. We will identify in this section the linear functional T_P with the polynomial P. Thus we may write $P \in (S,\tau)'$.

Denote with S_E and S_G the spaces spanned, as in the preceding section, by evaluations in points of E and of G, respectively. Then the inclusion map ι from E to G induces a map from S_E to S_G

$$\sum_{j=1}^n e_{x_j} \mapsto \sum_{j=1}^n e_{\iota(x_j)},$$

which is one-to-one, thanks to the Lemma in §1, and the Hahn-Banach theorem. Indeed, if $\sum_{j=1}^n e_{\iota(x_j)} = 0$ in S_G, and $\gamma \in E'$, then extend γ to $\Gamma \in G'$, and we have

$$\left(\sum_{j=1}^n e_{x_j}\right)(\gamma) = \left(\sum_{j=1}^n e_{\iota(x_j)}\right)(\Gamma) = 0$$

for any $\gamma \in E'$. Thus $\sum_{j=1}^{n} e_{x_j} = 0$. We will drop the ι in the sequel and consider any $s \in S_E$ an element of S_G.

Note that any $P \in (S_E)^*$ (the algebraic dual of S_E) extends to S_G, so any polynomial can be 'algebraically' extended, the real question is what type of polynomial we can expect this extension to be. Now consider locally convex, Hausdorff topologies τ_E and τ_G on S_E and S_G. Then the following proposition is immediate.

Proposition: Given the spaces $(S_E, \tau_E)'$ and $(S_G, \tau_G)'$ of polynomials over E and G respectively, then a polynomial $P \in (S_E, \tau_E)'$ extends to a polynomial in $(S_G, \tau_G)'$ if and only if it is τ_G-continuous (i.e. for the topology induced by τ_G on S_E).

We now extend integral polynomials over E to integral polynomials over an arbitrary larger space G.

Theorem: Any $P \in P_I({}^kE)$ can be extended to $\tilde{P} \in P_I({}^kG)$, with $\|\tilde{P}\|_I = \|P\|_I$.

Proof: Recall that $P_I({}^kE)$ is isometrically the dual of S_E where this space is viewed as a subspace of $P({}^kE')$, and similarly, $P_I({}^kG)$ is isometrically the dual of S_G, a subspace of $P({}^kG')$. But the map $S_E \longrightarrow S_G$ is simply the restriction of the morphism

$$P({}^kE') \longrightarrow P({}^kG') \qquad\qquad Q \mapsto \tilde{Q} \text{ such that } \tilde{Q}(\alpha) = Q(\rho(\alpha))$$

(where $\rho : G' \longrightarrow E'$ is the restriction map). This is an embedding of $P({}^kE')$ as a closed subspace of $P({}^kG')$:

$$\|\tilde{Q}\| = \sup_{\alpha \in B_{G'}} |\tilde{Q}(\alpha)| = \sup_{\alpha \in B_{G'}} |Q(\rho(\alpha))| = \sup_{\gamma \in B_{E'}} |Q(\gamma)| = \|Q\|.$$

Thus S_E is a subspace of S_G, and the result follows from the Hahn-Banach theorem.

We note that the converse of the theorem is not, in general, true. There are extendible polynomials which are not integral. This is, in fact, a corollary of the theorem. Recall [6] that a Banach space E is said to be symmetrically regular if every continuous symmetric linear mapping $T : E \longrightarrow E'$ is weakly compact. ℓ_1 is known to be non-symmetrically regular, while $C([0,1])$, for example, is symmetrically regular (a concrete example of a non-weakly compact operator $T : \ell_1 \longrightarrow \ell_\infty$ can be found in [4]).

Corollary: There is an extendible, non-integral polynomial over c_0.

Proof: Every polynomial over c_0 is extendible. In fact, if $c_0 \subset G$, ℓ_∞ is complemented in G'' (ℓ_∞ is complemented in any space that contains it), so there is an extension morphism for continuous homogeneous polynomials. But not every polynomial over c_0 is integral. Integrality and nuclearity coincide on c_0 by [1] for ℓ_1, being a separable dual, has the Radon-Nikodym property, thus we need only show that not every polynomial on c_0 is nuclear.

If every polynomial on c_0 were nuclear, $P_N({}^2c_0)$ would be isomorphic to $P({}^2c_0)$. Every polynomial on c_0 is approximable, so the space $P_A({}^2c_0)$ of approximable polynomials would be isomorphic to the space of nuclear polynomials $P_N({}^2c_0)$. Taking duals, one obtains, through the Borel transform [14], that all 2-homogeneous continuous polynomials on ℓ_1 are integral. By the Theorem above, every 2-homogeneous continuous polynomial on ℓ_1 would be extendible, but this is false. There are non-extendible polynomials on ℓ_1.

To see this, consider a non-weakly compact symmetric linear operator $T : \ell_1 \longrightarrow \ell_\infty$, and consider the inclusion $\ell_1 \subset C([0,1])$. T corresponds to a 2-homogeneous polynomial P over ℓ_1, and its extension to $C([0,1])$ would give rise to a continuous symmetric operator $S : C([0,1]) \longrightarrow C([0,1])'$ making the following diagram commutative.

$$\begin{array}{ccc} \ell_1 & \xrightarrow{T} & \ell_\infty \\ J \downarrow & & \uparrow J' \\ C([0,1]) & \xrightarrow{S} & C([0,1])' \end{array}$$

But this cannot be, for the symmetric regularity of $C([0,1])$ assures the weak compactness of S, and thus of T.

Many sub-classes (finite-type, nuclear) of integral polynomials over E can of course be extended to the corresponding kind of polynomial over G. We next prove this kind of correspondence for polynomials in $(S_E, \tau_0)'$ (see also [22]).

Proposition: Each $P \in (S_E, \tau_0)'$ extends to $\tilde{P} \in (S_G, \tau_0)'$.
Proof: P corresponds to a τ_0-continuous linear form over S_E. Thus there is a compact subset $K \subset E'$ and a constant $c > 0$ such that

$$|T_P(s)| \leq c \sup_K |s| \qquad \text{for all } s \in S_E.$$

Let $\rho : G' \longrightarrow E'$ be the restriction map. By Hahn-Banach this is a surjective map, so the Michael continuous selection principle [8] allows for the existence of a non-linear continuous section $\sigma : E' \longrightarrow G'$. Then $\sigma(K)$ is a compact subset of G' and

$$\sup_K |s| = \sup_{\gamma \in K} \left| \sum_{j=1}^{n} \gamma(x_j)^k \right| = \sup_{\gamma \in K} \left| \sum_{j=1}^{n} \rho\sigma(\gamma)(x_j)^k \right| = \sup_{\alpha \in \sigma(K)} \left| \sum_{j=1}^{n} \rho(\alpha)(x_j)^k \right| = \sup_{\sigma(K)} |s|.$$

Thus $|T_P(s)| \leq c \sup_K |s| = c \sup_{\sigma(K)} |s|$ for all $s \in S_E$. T_P is τ_0-continuous (for the τ_0-topology induced by S_G on S_E) and therefore extends by Hanh-Banach to $\tilde{P} \in (S_G, \tau_0)'$.

Note that in general, neither weakly continuous nor even approximable polynomials need be extendible. Kirwan and Ryan [18] have proved that the extendible polynomials over ℓ_2 are exactly the nuclear polynomials. But there are approximable polynomials over ℓ_2 which are not nuclear.

3 Aron-Berner extension of integral polynomials

Recall that in [3] Aron and Berner found a way to extend any continuous homogeneous polynomial from E to its bidual E'' (see also [9], [27]). This extension is in fact purely algebraic in nature and can be applied to any polynomial, continuous or not. It is easily verified that if P is a nuclear polynomial over E, then its Aron-Berner extension $\overline{P}$ is also nuclear. In fact, if $P = \sum_i \gamma_i^k$, then $\overline{P} = \sum_i \overline{\gamma_i}^k$, with $\overline{\gamma} = j(\gamma)$ where $j : E' \longrightarrow E'''$ is the canonical inclusion. If P is integral, and μ is a regular Borel measure on $B_{E'}$ representing P, then one is tempted to put, for $z \in E''$,

$$\overline{P}(z) = \int_{B_{E'}} z(\gamma)^k d\mu(\gamma).$$

The problem with this expression is that this integral may not exist, in fact z^k need not be a μ-measurable function. Note that z^k is not a continuous function on $(B_{E'}, w^*)$, nor is it the pointwise limit of a sequence (x_n^k) of powers of elements of E (which are known to be integrable).

We will prove that the validity of the above expression for the Aron-Berner extension of an integral polynomial is equivalent to E not containing an isomorphic copy of ℓ_1. For this we will use the equivalence given by Haydon in [16], and the characterization of the Aron-Berner extension in [27]. We will also be able to prove that the Aron-Berner extension of an integral polynomial is always integral, with the same integral norm, even when the above expression is not valid.

μ will denote a regular Borel measure on $(B_{E'}, w^*)$, and k a positive integer. Define $S : L_1(\mu) \longrightarrow E'$ as

$$S(f)(x) = \int_{B_{E'}} f(\gamma)\gamma(x) d\mu(\gamma)$$

and consider $S' : E'' \longrightarrow L_1(\mu)' = L_\infty(\mu)$. It easily follows that $\|S'\| = \|S\| \leq 1$ and, for $x \in E$, $S'(x) = \hat{x}$, where $\hat{x}$ is the class of the function $\gamma \mapsto \gamma(x)$.

Lemma: If P is an integral polynomial over E, represented by the measure μ, then its Aron-Berner extension $\overline{P}$ may be written

$$\overline{P}(z) = \int_{B_{E'}} S'(z)^k d\mu$$

Proof: Define the continuous k-homogeneous polynomial Q over E'' by

$$Q(z) = \int_{B_{E'}} S'(z)^k d\mu$$

Since Q is an extension of P, we can check that $Q = \overline{P}$ by proving that its first-order differentials at all points $z \in E''$ are w^*-continuous [27]. Indeed,

$$DQ(z)(w) = \int_{B_{E'}} S'(z)^{k-1} S'(w) d\mu$$

is a w^*-continuous function of w, for S', being a transpose, is $w^* - w^*$-continuous, and $S'(z)^{k-1}$ is in $L_1(\mu)$ because it is bounded on a finite measure space.

Corollary: The Aron-Berner extension of an integral polynomial is an integral polynomial and has the same integral norm.
Proof: A polynomial Q on a Banach space X is integral if and only if there exists a finite measure ν on a compact space Ω and a bounded linear operator $R : X \longrightarrow L_\infty(\Omega)$ such that

$$Q(x) = \int_\Omega R(x)^k d\nu$$

(see [10]). Moreover, we have $\|Q\|_I \leq \|R\|^k \ |\nu|$.

In our case, the previous lemma gives such a factorization and, since $\|S'\| \leq 1$, $\|\overline{P}\|_I \leq \|P\|_I$. The other inequality is a consequence of our first proposition of §1, since $\overline{P}$ is an extension of P as a linear functional.

In the following theorem, μ denotes any regular Borel measure on $(B_{E'}, w^*)$. Haydon [16] (see also [11]) has proved that a Banach space E does not contain a copy of ℓ_1 if and only if the elements of E'' are μ-measurable. Thus, z will denote both an element of E'' and its class in $L_\infty(\mu)$.

Theorem: Let E be a Banach space. Then the following are equivalent:

- i) E does not contain a copy of ℓ_1.
- ii) For all μ, the map $E'' \longrightarrow L_\infty(\mu)$ $(z \mapsto z)$ is well defined and $w^* - w^*$-continuous.
- iii) For all μ, and all $z \in E''$, $S'(z) = z$ μ-a.e.
- iv) For all μ, each $z \in E''$ is μ-measurable and the Aron-Berner extension of $P(x) = \int \gamma(x)^k d\mu$ is

$$\overline{P}(z) = \int_{B_{E'}} z(\gamma)^k d\mu(\gamma)$$

QUAERERE VERUM Universidad de San Andrés

Proof: i) implies ii): By Haydon's theorem [16], the mapping in ii) is well-defined. We need to show that for any $f \in L_1(\mu)$, the linear map T_f defined by

$$T_f(z) = \int_{B_{E'}} z f d\mu$$

is w^*-continuous. To this end, note that $fd\mu$ is a regular Borel measure, which will be assumed to be a probability measure (for the general case, just write $fd\mu$ as a linear combination of probability measures). Again by [16], every $z \in E''$ satisfies the barycentric calculus, so if $\gamma \in B_{X'}$ is the barycenter of $fd\mu$, we have

$$T_f(z) \;=\; \int_{B_{E'}} z f d\mu \;=\; z(\gamma)$$

Thus T_f is w^*-continuous.

ii) implies iii): For any $f \in L_1(\mu)$, we wish to see that

$$\int_{B_{E'}} S'(z) f d\mu = \int_{B_{E'}} z f d\mu$$

The equation holds for all $z \in E$, and both sides are w^*-continuous functions of z (the left side is always w^*-continuous, the right side by ii)). Thus iii) follows from Goldstine's theorem.

iii) implies iv): iii) and the previous lemma immediately imply iv).

iv) implies i): i) is, once again, a consequence of Haydon's theorem.

Universidad de San Andrés QUAERERE VERUM

References

[1] R. Alencar, *Multilinear mappings of nuclear and integral type*, Proc. Amer. Math. Soc. **94** (1985), 33-38.

[2] R. Alencar, *On reflexivity and basis for $P(^mE)$*, Proc. R. Ir. Acad. **85A, No. 2** (1985), 131-138.

[3] R. Aron and P. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bull. Soc. Math. France **106** (1978), 3–24.

[4] R. Aron, B. Cole, and T. Gamelin, *Spectra of algebras of analytic functions on a Banach space*, J. reine angew. Math. **415** (1991), 51–93.

[5] R. Aron and P. Galindo, *Weakly compact multilinear mappings*, (to appear).

[6] R. Aron, P. Galindo, D. García, and M. Maestre, *Regularity and algebras of analytic functions in infinite dimensions*, Trans. of the Amer. Math. Soc. **348(2)** (1996), 543–559.

[7] R. Aron, M. Lindström, and R. Ryan, *Uniform factorization for compact sets of operators*, (to appear).

[8] R. Bartle and L. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc. **72** (1952), 400–413.

[9] A. Davie and T. Gamelin, *A theorem on polynomial-star approximation*, Proc. Amer. Math. Soc. **106** (1989), 351-356.

[10] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, Math. Studies 176, North-Holland, Amsterdam (1993).

[11] J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, 92, Springer-Verlag, New York (1984).

[12] S. Dineen, *Complex Analysis in Locally Convex Spaces*, Math. Studies 57, North-Holland, Amsterdam (1981).

[13] S. Dineen and R. Timoney, *Complex geodesics on convex domains*, Progress in Functional Analysis (ed. Bierstedt, Bonet, Horváth, Maestre), Math. Studies 170, North-Holland, Amsterdam (1992), 333-365.

[14] C. Gupta, *Malgrange theorem for nuclearly entire functions of bounded type on a Banach space*, Ph.D.Thesis, I.M.P.A., Rio de Janeiro and University of Rochester (1966).

[15] P. Galindo, D. García, M. Maestre, and J. Mujica, *Extension of multilinear mappings on Banach spaces*, (to appear).

[16] R. Haydon, *Some more characterizations of Banach spaces containing* ℓ_1, Math. Proc. Camb. Phil. Soc. **80** (1976), 269-276.

[17] J. A. Jaramillo, A. Prieto, and I. Zalduendo, *The bidual of the space of polynomials on a Banach space*, Math. Proc. Camb. Phil. Soc. (to appear).

[18] P. Kirwan and R. Ryan, *Extendibility of homogeneous polynomials on Banach spaces*, (to appear).

[19] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. **48** (1964).

[20] M. Lindström and R. Ryan, *Applications of ultraproducts to infinite dimensional holomorphy*, Math. Scand. **71** (1992), 229-242.

[21] P. Mazet, *A Hahn-Banach theorem for quadratic forms*, (to appear).

[22] L. Moraes, *A Hahn-Banach extension theorem for some holomorphic functions*, Complex Analysis, Functional Analysis and Approximation Theory (ed. J. Mujica), Math Studies 125, North-Holland, Amsterdam (1986), 205-220.

[23] J. Mujica, *Complex Analysis in Banach Spaces*, Math. Studies 120, North-Holland, Amsterdam (1986).

[24] J. Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **324 (2)** (1991), 867–887.

[25] R. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Ph.D.Thesis, University College, Dublin (1980).

[26] E. Toma, *Aplicacoes holomorfas e polinomios* τ*-continuos*, Thesis, Universidade Federal do Rio de Janeiro (1993).

[27] I. Zalduendo, *A canonical extension for analytic functions on Banach spaces*, Trans. Amer. Math. Soc. **320(2)** (1990), 747–763.

Departamento de Economía y Matemática
Universidad de San Andrés
Vito Dumas 284
(1644) Victoria
ARGENTINA

daniel@udesa.edu.ar
nacho@udesa.edu.ar

QUAERERE VERUM
Universidad de San Andrés