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# *Dynamic Bank Runs*


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## "Corridas Bancarias Dinámicas"

Resumen

*Este trabajo presenta un modelo de corridas bancarias dinámicas. En contraposición con modelos de horizonte finito, basados en Diamond y Dybvig (1983), en los que las corridas bancarias tienen perfecta sincronización e involucran a todos los agentes, encuentro que existen contextos que permiten corridas parciales en los que únicamente una fracción de los depositantes retiran. Más aún, existen equilibrios con corridas parciales en los que la corrida termina en una cantidad finita de períodos, y otros en los que hay depositantes que retiran en todos los períodos. Aunque dada una disposición de parámetros el modelo admita múltiples equilibrios, encuentro que hay una relación entre el tipo de corrida que puede ocurrir en equilibrio y características de la economía que determinan la solidez de la hoja de balance del banco. Finalmente, presento microfundamentos alternativos para la demanda de seguro de liquidez, con dos conclusiones. Primero, que una amplia clase de arreglos institucionales admiten fenómenos similares a corridas bancarias, incluyendo contratos diseñados con compatibilidad de incentivos que usualmente se consideran invulnerables. En segundo lugar, este trabajo muestra con un ejemplo que, aun cuando sean relevantes en casos específicos, los costos de liquidación temprana no son un componente esencial de las corridas bancarias.*

Palabras clave: corridas bancarias, coordinación, expectativas, costos de liquidación temprana, compatibilidad de incentivos



## "Dynamic Bank Runs"

Abstract

*I develop a model of bank runs that is dynamic from the point of view of the individual depositor. As opposed to finite-horizon models based on Diamond and Dybvig (1983), where runs are synchronized and involve all agents, I find that some contexts allow for partial runs, in which only a fraction of depositors withdraw, and that other contexts have the massive run as the only equilibrium run. Further, partial runs can last finitely many periods or go on* *forever. Although the model has multiple equilibria, I show that there is a relation between which kind of run can arise in equilibrium and fundamentals and the solidness of the bank's balance sheet. I also introduce an alternative microfoundation for the demand of a liquidity insurance, with two insights. First, that runs can emerge in a broad class of arrangements usually deemed invulnerable (because they contain an incentive-compatibility clause), such as dynamic contracts with limited commitment. Second, that early liquidation costs, although relevant and important, are not an essential component of bank runs.*

Keywords: bank runs, coordination, expectations, early liquidation costs, incentive compatibility

Códigos JEL: E5, G21, G28

# 1 Introduction[1]

During the past decades, substantial and fruitful work[2] has been devoted toward understanding banking crises, their causes, and institutional arrangements that could prevent them. The dynamic nature of such crises, however, has been ignored. This paper develops a model of banking and bank runs that is dynamic from the point of view of the individual depositor. In the model, some (but not all) contexts[3] will generate massive synchronized instantaneous runs –like the one originally depicted by Diamond and Dybvig (1983). In turn, other contexts will generate partial runs, in which only a fraction of the depositors withdraw. In turn, such partial runs can either last finitely many periods or go on forever, depending on expectations in a self-fulfilling manner.

The vast literature on bank runs, rooted in the Diamond-Dybvig (DD) model, has started by establishing that the mismatch between the maturity structure of assets and liabilities is both the core of the run threat and also the fundamental reason why there are banks at all.

I argue that the demand for a liquidity insurance can be modelled -and microfounded- in a standard way by using income shocks instead of preference shocks. Doing so is useful in the sense that it allows a simple tractable dynamic analysis. In my model, the bank is a moneylender in the exact spirit of Phelan (1995) or Kocherlakota (1996), interpreting lack of commitment as the depositors' ability to quit the bank[4] at any time. Specifically, agents in the model are endowed with a random consumption stream, and this generates demand for insurance. Insurance is provided by the bank, with the caveat that agents are unable to pre-commit to remain in the contract forever. Therefore, the bank designs the contract to make it incentive-compatible that every insured agent only consumes the amount prescribed by the contract. However, I show that there are equilibria (in the post-deposit game) in which at least some agents effectively deviate from the contract. In those cases, we will say that a run is taking place.

This new interpretation of the liquidity mismatch allows for a straightforward extension of bank runs analysis to more general settings. I argue that runs can arise in any context in which an attempt to hedge risk is undertaken by agents that are not forcefully bound to keeping their promises. Notably, the class of dynamic contracts with limited commitment falls within this category. Because such contracts include an incentive-compatibility constraint, they are usually regarded as being invulnerable to runs. However, since the bank in my model offers exactly one such contract, this paper shows that they are not. The reason is that, as Peck and Shell (2003) point out, the incentive-compatibility constraint presupposes that no agents are deviating.

The idea of extending bank runs analysis to other institutional arrangements has already been advanced. For instance, He and Xiong (2012) show that rollover risk may induce creditors to run on a private firm. In fact, in their paper, a global-games modelling allows them to show that the only equilibrium features a run whenever the fundamentals enter a critical region. Brunnermeier and Pedersen (2009) also describe a run on assets in which the ignorance of some speculators about the true state of the economy induces them to fire-sell their assets when they are still valuable. One contribution of my model is thus to take the idea further, as I show that runs can occur in contexts apparently very different from deposit contracts, and which are also generally regarded as being invulnerable to runs.

The modelling of the bank and the bank run presented here does not assume capital losses from early liquidation. Therefore, since runs indeed happen, liquidation costs are not essential to the occurrence of bank runs. Rather, although such costs exist and are relevant, their inclusion in bank

[1]I am indebted to my advisor Enrique Kawamura, Daniel Heymann, Martn Gonzalez-Eiras and Juan Carlos Hallak for helpful discussions and wise comments. All remaining errors are my own.

[2]for general surveys, see Allen and Gale (2009) and Freixas and Rochet (2008)

[3]by contexts I mean combinations of institutional arrangement and features of the environment and of the depositors' preferences, notably, their degree of risk-aversion

[4]in the model, this will be the only "excessive" withdrawal available

runs models responds to technical needs that arise from the particular microfoundation, namely, that the demand for liquidity insurance comes from preference shocks. When income is fixed *ex-ante*, no trade is possible between the agents in absence of investment opportunities, since an early consumer would have nothing to offer a late consumer in exchange for goods today. With income shocks, an "early consumer," that is, an agent with a low current realization, who would want to borrow in a complete markets environment, still has her future endowment to offer in exchange for more consumption today. This is what the bank in my model ultimately exploits. As a consequence, early liquidation costs are never a part of setup: runs are a matter of trust alone, rooted only in the sequential service constraint.

Many features of bank runs naturally present a dynamic nature. It has been argued that banking crises are deeply related to currency crises.[5] However, many ideas from the currency crises literature are absent from the theory of bank runs. For instance, Krugman (1979) shows that a currency run has a dynamic profile, in which reserves are slowly depleted until a massive synchronized speculative attack forces the Central Bank to quit the exchange target. Many episodes of banking runs share this profile,[6] and the framework I develop can help explain it.

I show that certain contexts allow for the occurrence of partial runs, in which only a fraction of the agents withdraw. Partial runs can last finitely many periods or go on forever, depending on expectations. In such runs, agents with higher current income realizations will quit the bank, while agents with lower realizations still demand insurance. The possibility of partial runs is closely related to regulation regarding bank bankruptcy: they only arise when the bank is allowed to continue operating after it has failed to honor some of its obligations. On the other hand, I show that, when the bank is destroyed after any failure, the only run that can arise in equilibrium is the massive run.

In a partial run that lasts many periods, the agents will plan to wait until they reach a certain state to withdraw. This decision will be subtle since remaining when a run is under way provides insurance but also entails positive probability of zero current consumption. Therefore, as an agent is uncertain about her future income shocks, remaining until a certain shock arrives will imply trading one risk for another: her own risky income stream against imperfect insurance that includes the possibility of being forced to exit the contract at some future (random) time and consume nothing in that period.

This paper is closely related to Gertler and Kiyotaki (2012). They also explore the link between bank runs and fundamentals in a dynamic setup. In their model, moral hazard concerns limit how much capital bankers can raise. This condition combined with an endogenously determined liquidation price (given by the public's ability to operate the capital themselves, with higher costs) determines whether a run can happen in equilibrium.

Other applications of dynamic considerations to the issue of bank runs have been made. Qi (1994) shows that, in an infinitely repeated DD environment with overlapping generations, there is a role for the bank even if side trades are allowed. He also shows that runs arise not only because of withdrawals but also because of lack of new deposits. This, in turn, implies that suspension of convertibility might not be effective to prevent runs.

The idea that early liquidation costs are not essential to bank runs is also related to Carmona (2004), who shows that, in the original DD framework with aggregate uncertainty, there is a positive probability of a bank run even without liquidation costs, in pre-deposit equilibrium. On a similar note, Allen and Gale (2004) endogenize the liquidation costs, and obtain that runs still occur –although they can also be efficient. In any case, the run still implies liquidation, and although the original investment might be recovered in full, liquidation still implies a loss from forgone future profits. In my model, as the bank solely redistributes current income, runs arise in the absence of liquidation.

Following Peck and Shell (2003), we reconcile rational expectations and the possibility of a crisis by analysing the post-deposit game. An alternative interpretation is that we assume that, for contract

[5] see Allen and Gale (2004), or Kawamura (2007), Chang and Velasco (2001) and Kaminsky and Reinhart (1999)

[6] an example is the 2001 Argentinian banking and currency crisis. In that episode, deposits had been falling throughout the year, but the rate of withdrawals spiked on November 29, see Ennis and Keister (2009)

writing and depositing purposes the agents attach zero probability to this event. This is what happens in, for example, Gertler and Kiyotaki (2012). Other possibilities have been explored in other contexts.[7] Goldstein and Pauzner (2005) use sunspots to relate probability of a run to *ex-ante* welfare, while Geanakoplos (2010) argues that assets fire sales can arise when new information dramatically changes who is willing to hold the asset. These kinds of assumptions can be introduced into this model; however, the bank's reaction is likely to be important. The stress here is that the bank offers the constrained-efficient contract, which also ensures that a runs-free equilibrium exists.

The rest of the paper is organized as follows. Section 2 describes the setup of the model and the optimal contract offered by the bank. Sections 3 and 4 explore different assumptions about what happens to the bank when this contract is broken and present the main results. Finally, Section 5 concludes, while the results' proofs are gathered in the Appendix.

## 2 The Model

Time is discrete and goes on forever. There is one consumption good, and the economy is populated by a continuum of identical agents, with names in the unit interval. Each of these agents is endowed with a random stream of the good, and orders (stochastic) consumption streams $\{c_t\}_{t=\tau}^{\infty}$ according to $\mathbb{E}_\tau \sum_{t=\tau}^{\infty} \beta^{t-\tau} u(c_t)$, where $\mathbb{E}_\tau$ is the time-$\tau$ conditional expectation operator and $u$ is increasing, strictly concave, twice continuously differentiable, and satisfies Inada conditions.

We will also assume that the consumption good is perishable: no agent in the economy will have access to a storing technology. The endowment process $\{y(t)\}_{t=0}^{\infty}$ is assumed to be a real-valued, non-negative Markov process with a finite state-space $\mathcal{S} = \{y_1, \ldots, y_S\}$ and transition matrix $P = \{p_{ij}\}_{i,j\in\mathcal{S}}$. Also, we will adopt the convention that $y_{i+1} > y_i \geq 0$. We will denote the measure of agents in state $s$ at time $t$ by $\mu_s(t) = \mathbb{P}(y(t) = y_s)$. Then, we have $\mu_s(t+1) = \sum_{i\in S} \mu_i(t) p_{is}$. Notice that there is only idiosyncratic uncertainty: the aggregate variables are deterministic functions of the endowment distribution.

### 2.1 Autarky

Under non-storability, the autarkic allocation is very simple. Each period, every agent consumes her own contemporary income. The value of being in autarky in state $s$[8] is thus

$$v_a(\delta_s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(y(t)) \mid y(0) = y_s\right]. \tag{2.1}$$

If $\lambda$ is a probability measure, we will mean by $v_a(\lambda)$ the expected value of being in a state randomly chosen under $\lambda$. Applying Fubini's Theorem yields

$$v_a(\lambda) = \sum_{i\in S} v_a(\delta_i)\lambda_i.$$

In general, it not obvious *a priori* which is the best state to be in autarky. Expanding the expectation in Equation (2.1) and substracting the values for two different states gives

$$v_a(\delta_s) - v_a(\delta_i) = u(y_s) - u(y_i) + \beta \sum_{l\in\mathcal{S}} (p_{sl} - p_{il}) v_a(\delta_l).$$

[7] for a survey of macroeconomic models with financial frictions, see Brunnermeier et al. (In Press)

[8] we let $\delta_s$ denote Dirac's delta measure at state $s$, and we use the strong Markov property to condition on time-0 instead of time-$t$

The sign of this expression will depend, of course, on which state provides the highest income, but it will also depend on the transition probabilities. This is very intuitive: an agent will prefer a lower income today if this implies a sufficiently better perspective in the future.

### 2.2 The bank's plan

This economic setup is very simple and it is clear that the autarkic allocation fails to exploit gains from trade and risk-sharing between the agents.

Under competition, a (representative) bank will maximize expected consumer welfare, earning no profits. The bank's arrangement will be as follows: every period, each agent will pay an insurance fee, and the bank will offer in return a payment contingent on that agent's current income.

The maturity mismatch will be represented by the assumption that every agent can choose to quit the contract at any time, by refusing to pay the fee. The bank will, in this case, exert the punishment that the agent be cast into autarky for all subsequent periods. From Phelan (1995), we know that this threat can sustain a runs-free equilibrium in which constrained-efficiency is attained.

In the model, we will take the equivalent approach of assuming that each agent gives up her income stream in exchange for a contingent consumption stream chosen by the bank. We maintain the assumption that agents cannot commit to remain in the contract: should an agent decide to deviate, she will simply revert to autarky.

Denote the value of the bank's problem, when aggregate state is $\mu$, by $V_c(\mu)$, and the value of a depositing agent's problem, when individual state is $s$, by $v_c(\delta_s, \mu)$. We have

$$V_c(\mu) = \max_{c_s} \sum_{s\in\mathcal{S}} \mu_s u(c_s) + \beta V_c(\mu P)$$

$$\text{subject to } u(c_s) + \beta v_c(\delta_s P, \mu P) \geq u(y_s) + \beta v_a(\delta_s P)$$
$$\textstyle\sum_s \mu_s c_s \leq \sum_s \mu_s y_s$$

This bank just maximizes a weighted average of utility, subject to feasibility and an incentive-compatibility/participation constraint that ensures that, in each state, the agent has no incentives to deviate from the contract. This implies that there will always be a runs-free equilibrium.

Attaching multipliers $\lambda_s$ and $\eta$ to the constraints, we find the first order conditions with respect to contingent consumption to be

$$u'(c_s)\frac{\mu_s + \lambda_s}{\mu_s} = \eta \geq 0 \tag{2.2}$$

where $\eta = 0$ if and only if the feasibility constraint is not binding. However, since the (competitive) bank attempts to increase welfare and the good is perishable, feasibility always binds.

Observe also that if no participation constraint binds (this property will depend on the stochastic process for income and the utility function), then the bank achieves the first-best allocation by giving agents in all states the same level of consumption. Furthermore, when the participation constraint is not binding on two or more states, then consumption will be the same on those states. On the other hand, we have the well-known result (see Ljungqvist and Sargent, 2004, chap. 19) that $\lambda_s$ is an increasing function of the optimal consumption level $c_s$.

## 3 Banking failure

Now, assume that the bank has already been created, according to the formulation from Section 2.2. Following Peck and Shell (2003), we will explore the post-deposit game.

Our setting differs from Diamond and Dybvig's in the following ways. First, we have $S$ instead of two types of agents, depending on their current endowment realization. Second, each type of agent will want to withdraw in different scenarios. Third, and this is the crucial difference, to assess the appeal of a particular strategy, an agent needs to know not only how many other agents are withdrawing but also who they are.

We are interested in characterizing what kind of runs can happen in equilibrium, and also in finding belief systems that imply optimality of a run for all or some agents. In the original Diamond-Dybvig framework, this problem is closed. The only run is the massive run, since any belief system that implies optimality of a run for some agent also implies its optimality for everyone else. The set of all these belief systems, moreover, consists of the expectations under which at least a fraction $f = (c_1^{1\star})^{-1}$ of agents is going to withdraw.

As in Diamond and Dybvig (1983), the sequential service constraint will occupy the core of the model. According to this constraint, agents who want to withdraw get in line and receive their payments until the bank's funds are exhausted. However, the setup of this model makes the workings of this restriction a little different. We assume that the sequential service constraint is operative at all times, so that agents that want to consume the amount $c_s$ specified in the contract are subject to the restriction. On the other hand, deviating agents withdraw at the beginning of the period. Since, in this model, the bank's resources are by definition just enough for everyone to deviate from the contract, deviating agents will not be affected by the sequential service constraint in equilibrium. In the DD model, withdrawing agents face the constraint with positive probability during a run, whereas remaining agents face it with probability 1 (the following period); in our setting, agents are ordered in the same way, as remaining agents face the constraint with positive probability while withdrawing agents face it with probability zero.

In this model, a deviating agent never gets in line, she just refuses to give the bank her endowment at the beginning of the period and then reverts to autarky. Thus, in the event of a run, if a fraction $f_s$ of type-$s$ agents decide to deviate (withdraw), the bank will be able to keep its promise if $\sum_s \mu_s(1 - f_s)c_s \leq \sum_s \mu_s(1 - f_s)y_s$, and will be in trouble otherwise. Should the bank fail, it will pay a fraction $\frac{\sum_s \mu_s(1-f_s)y_s}{\sum_s \mu_s(1-f_s)c_s}$ of its obligations, and will subsequently be destroyed at the end of the period. In equilibrium, we will assume that the distribution of agents in line is perceived *ex-ante* as uniform; therefore, $\frac{\sum_s \mu_s(1-f_s)y_s}{\sum_s \mu_s(1-f_s)c_s}$ will also be the probability with which a remaining agent will expect to be paid her under-the-contract consumption.



### 3.1 First-order runs

When an agent expects a fraction $f_s$ of type-$s$ agents to withdraw, her participation constraint is changed beyond the committed bank's control. The agent then needs to assess whether to stay or to withdraw. Let $A$ stand for the event in which the bank survives. That is, let $A = \{(f_s)_s : \sum_s \mu_s(1 - f_s)c_s \leq \sum_s \mu_s(1 - f_s)y_s\}$. Withdrawing will be preferred iff

$$\mathbf{1}_A \left[ u(c_s) + \beta v_c \left( \delta_s P, \left( \frac{\mu_s(1-f_s)}{\sum_s \mu_s(1-f_s)} \right)_s P \right) \right] + \mathbf{1}_{A^c} \left[ u(c_s) \frac{\sum_s \mu_s(1-f_s)y_s}{\sum_s \mu_s(1-f_s)c_s} + \beta v_a(\delta_s P) \right] \leq \leq u(y_s) + \beta v_a(\delta_s P).$$

The left-hand side of this condition reflects the value of staying in the contract: if the bank does not fail, the agent obtains consumption $c_s$ now and the continuation value $v_c$ later, where the state variables move according to the agent's expectations; whereas if the bank fails, the agent obtains the continuation value given by autarky, and only obtains consumption $c_s$ with a probability that depends on the total amount that will be withdrawn. On the other hand, the value of leaving the contract is autarky starting from the current state, which we also write in terms of the current consumption, $y_s$, and the continuation value, $v_a(\delta_s P)$. The condition for withdrawing can be shown to be equivalent to

$$u(c_s)\left[\mathbf{1}_A + \mathbf{1}_{A^c}\frac{\sum_s \mu_s(1-f_s)y_s}{\sum_s \mu_s(1-f_s)c_s}\right] + \mathbf{1}_A\beta v_c\left(\delta_s P, \left(\frac{\mu_s(1-f_s)}{\sum_s \mu_s(1-f_s)}\right)_s \cdot P\right) \leq u(y_s) + \beta v_a(\delta_s P)\mathbf{1}_A. \quad (3.1)$$

or

$$u(c_s) + \mathbf{1}_A\beta v_c\left(\delta_s P, \left(\frac{\mu_s(1-f_s)}{\sum_s \mu_s(1-f_s)}\right)_s \cdot P\right) \leq u(y_s)\left[\mathbf{1}_A + \mathbf{1}_{A^c}\frac{\sum_s \mu_s(1-f_s)c_s}{\sum_s \mu_s(1-f_s)y_s}\right] + \beta v_a(\delta_s P)\mathbf{1}_A \quad (3.2)$$

There are some conclusions to be drawn from the examination of Equation (3.1) from an individual's point of view.

First of all, as follows from the contract design, no agents deviating is an equilibrium: if $f_s = 0$ for each $s$ (notice that the bank survives), then remaining in the contract will be at least as good as withdrawing iff

$$u(c_s) + \beta v_c(\delta_s P, \mu P) \geq u(y_s) + \beta v_a(\delta_s P),$$

which is exactly the participation constraint. Therefore, also, only those agents with a binding participation constraint can deviate from the contract without believing that other agents are also withdrawing. Note, however, that the expectation that other agents are withdrawing can potentially lead any agent to withdraw, even those for whom the participation constraint is not binding.

Second, the bank is immune to scale-runs, that is, there is no equilibrium in which the same fraction of each type of agent deviates. The reason is that, again, setting every $f_s = f$ implies that $A$ is a set of probability one. Equation (3.1) then yields the participation constraint, which is satisfied by the bank's plan.

In Equation (3.1), the value of staying in the contract depends on the distribution of agents remaining in the contract the following period. This fact reflects a novel feature of the dynamic setting. An agent's optimal withdrawal choice depends on her expectations not only through her expectations of the bank failing but also through her expectations of the future composition of the bank's balance sheet. When the fraction $\{f_s\}$ withdraws, both the bank's weighting system and its future income depend on the new measure $\left(\frac{\mu_s(1-f_s)}{\sum_s \mu_s(1-f_s)}\right)_s$. In a static version of the model, all that matters is the total amount withdrawn. Now we also need to consider the way in which the withdrawals shift the bank's probability measure.

### 3.1.1 A two-state income process

In this section, we will assume that the income process can only take up two values: $y_2 > y_1 \geq 0$. In this case, we have a basic fact about the participation constraints pattern.

**Lemma 3.1.** *Either no participation constraint binds, and then $c_1 = c_2$, or it binds for the high state, and $c_2 > c_1$.*

For simplicity, we will concentrate on the case in which the high-state participation constraint binds. To characterize the equilibria with runs in the two-state model, first observe that the type-2 agents are indifferent between staying in the contract and deviating. Thus, it is consistent for a type-1 agent to expect them to act in either direction.

Now suppose that a type-1 agent expects at least a proportion $1 - \alpha$ of type-2 agents to withdraw, and in principle expects all type-1 agents to stay. Since, in the two-state case, the bank redistributes resources from type-2 to type-1 agents, these expectations imply that the bank will fail. The agent needs to decide whether to get in line to receive $c_1$ (if she gets to the bank "soon enough") or to deviate from the contract and withdraw $y_1$. In either scenario, the agent's expectation is that the bank will fail, so the continuation value is $\beta v_a(\delta_1 P)$. Therefore, according to (3.1), the type-1 agent withdraws iff

$$u(c_1)\frac{\mu_1 y_1 + \mu_2 \alpha y_2}{\mu_1 c_1 + \mu_2 \alpha c_2} \leq u(y_1),$$

which is equivalent to

$$\alpha \leq \frac{\mu_1}{\mu_2}\frac{c_1 u(y_1) - y_1 u(c_1)}{y_2 u(c_1) - c_2 u(y_1)} = \alpha^\star \tag{3.3}$$

Equation (3.3) shows that expectations leading to a run exist iff $\alpha^\star > 0$. Since the denominator is positive ($y_2 > c_2$ and $c_1 > y_1$), the necessary and sufficient condition for existence is that $c_1 u(y_1) > y_1 u(c_1)$, or equivalently, that $\frac{c_1}{y_1} > \frac{u(c_1)}{u(y_1)}$. This is just a condition on the risk-aversion: the inequality will be satisfied by any strictly concave utility function.

**Proposition 3.2.** *If the bank is destroyed after it fails, and the participation constraint binds for type-2 agents, the only possible run is a massive run.*

Therefore, we have that any degree of risk-aversion[9] implies that two equilibria are present in the two-state case: that everyone runs and that no one does.

### 3.1.2 The general case

We will now lift the assumption that the income process needs to take up only two values. In this general case, as we have already seen, an agent in autarky might prefer a lower contemporaneous income in exchange for better prospects. The agent will thus order states in a way that reflects not only current income but also the transition probabilities.

Therefore, in the general case, Lemma 3.1 will no longer hold. However, as we will see, we will be able to partition the agents into two broad classes. In turn, these classes will bear some resemblance to the type-1 and type-2 agents in the two-state case.

We will define the following classes. Let state $s$ be of class 2 if the optimal contract's participation constraint is binding at state $s$. Conversely, let state $s$ be of class 1 if the optimal contract's participation constraint does not bind at $s$. Notice that, from the bank's problem FOC (2.2), every state of class 1 gets the same consumption level, $c^{(1)}$.

**Theorem 3.3.** *If the bank is destroyed after it fails, and the participation constraint binds at least at one state under the optimal contract, the only possible run is a massive run.*

The argument is roughly the same as the two-state case: when they expect the bank to fail, all participation-constrained agents will withdraw. This induces a cascade of withdrawals by unconstrained agents who expect others to withdraw. Therefore, as in the two-state case, assuming that the bank is destroyed after failure yields the result that only two equilibria are possible: either everyone runs or no one does.

## 3.2 Higher-order runs

So far, we have asked ourselves what kinds of behavior are compatible with equilibrium when agents expect other people to withdraw on the contemporaneous period. In this section, we will address the question of what can happen when agents anticipate others to withdraw in the future.

Anticipation of a run in the future will lead to a run today or to nothing at all, depending on a technical condition. To see this, consider the strategies of an agent whose participation constraint is

[9] remember that, in the model, risk-aversion is the only source of the demand for a liquidity insurance contract. Really, this fact is consistent with Diamond and Dybvig's main finding, that the liquidity mismatch is both the reason why banks exist and the reason why they are vulnerable.

binding under the contract today, an agent in a state of class 2. If a run is going to take place at some point in the future, we know it is going to be a massive run (since no other runs are possible). If the probability of entering a state of class 1 at any point in the future is positive,[10] then the future run (and the subsequent disappearance of the bank) lowers the value of remaining in the contract for this agent.

Since she was indifferent between remaining and withdrawing when expecting the contract to go on forever, this leads her to withdraw today. Now, every other agent can anticipate this reasoning. Thus, anticipating a run (anywhere) in the future leads to anticipating a run today. As massive runs are the only kind of run in this model, expecting a run in the future will lead to a massive run today.

All the results derived in this section depend strongly on the assumption that the bank is to be destroyed after any failure. In the next section, we will relax this assumption to allow some agents to remain in the contract after a run.

# 4 Runs on a surviving bank

In the preceding section, we showed that the bank was too static to allow for runs other than a massive synchronized run. In this section, we will allow the bank some flexibility that will give rise to dynamic runs. Indeed, as it stands out from the proof of Theorem 3.3, a significant amount of runs are triggered when agents are forced to weigh off the possibility of not getting in line early enough against benefits just on the current period, $u(c_s) - u(y_s)$, with autarky as the continuation value in any case. In this section, we will essentially give these agents something to compare deviation to.

The outline is simple: in the two-state case, a type-1 agent facing all type-2 agents withdrawing could consume $y_1$ with certainty or $c_1$ with a probability $p$ that satisfied $pc_1 = y_1$. This consumer was facing a mean-preserving spread, so the only thing that could induce her to remain in the contract was the promise of a continuation value. This explains why risk-aversion was a necessary and sufficient condition for type-1 agents to withdraw when facing all the type-2 agents withdrawing.

Until now, we maintained the assumption that the bank was destroyed whenever it could not honor its debts. In this section, we will assume that an agent who gets in line and receives her payment *can still* remain in the contract next period. That is, we are assuming that the contract binds each individual agent to the bank: whenever the agent receives her payment, there is no violation of the contract and the agent should be allowed to remain.

Hence, the expected value of staying in the contract when expectations imply that the bank will fail is now given by $u(c_s) + \beta v_c\left(\delta_s P, \left(\frac{\mu_s(1-f_s)}{\sum_s \mu_s(1-f_s)}\right)_s P\right)$, the current payment and continuation value, times the probability of getting paid: $\frac{\sum_s \mu_s(1-f_s)y_s}{\sum_s \mu_s(1-f_s)c_s}$. Notice that, since the place in line is independent of the state -conditional on being in line-, the bank's measure for next period equals the measure of agents remaining. Therefore, Condition (3.2) becomes to withdraw whenever

$$u(c_s) + \beta \mathbf{1}_A v_c\left(\delta_s P, \left(\frac{\mu_s(1-f_s)}{\sum_s \mu_s(1-f_s)}\right)_s P\right) \leq u(y_s)\left[\mathbf{1}_A + \mathbf{1}_{A^c}\frac{\sum_s \mu_s(1-f_s)c_s}{\sum_s \mu_s(1-f_s)y_s}\right] + \beta \mathbf{1}_A v_a(\delta_s P) \quad (4.1)$$

Obviously, when event $A$ occurs (when the bank survives), this condition remains unchanged by our new assumptions. However, when the bank cannot face all its liabilities, the expression is changed, because now a nondeviating agent does not face autarky with probability 1. Rather, she needs to consider the possibility of arriving in time to get paid and continue in the contract.

Assume once again the 2-state income process, and that a fraction $1-\alpha$ of type-2 agents are

[10] that is, if the set of class-2 states is not an absorbing set

withdrawing. The optimal reaction (4.1) for type-1 agents is to withdraw if

$$u(c_1) + \beta p_{11} v_c \left( \delta_1, \frac{(\mu_1, \alpha\mu_2)}{\mu_1 + \alpha\mu_2} \cdot P \right) \leq u(y_1) \frac{\mu_1 c_1 + \alpha\mu_2 c_2}{\mu_1 y_1 + \alpha\mu_2 y_2} + \beta p_{11} v_a(\delta_1), \tag{4.2}$$

where we have used the fact that the participation constraint on state-2 is binding.[11]

**Lemma 4.1.** *In the two-state case, every run will feature the bank failing and all the type-2 agents withdrawing.*

With this observation in mind, we can focus on what happens after every type-2 agent has deviated from the contract. We will show that, depending on parameters of the utility function and the income process, there can be a massive or a *partial* run.

In the *iid* case, $P$ has all rows equal to the measure $\mu$ (so that $\mu P = \mu$ and $\delta_s P = \mu$, for every $s$), and if we set $\alpha = 0$, (4.2) becomes to withdraw if

$$u(c_1) - u(y_1)\frac{c_1}{y_1} \leq \beta\mu_1 (v_a(\delta_1) - v_c(\delta_1, \mu)) \tag{4.3}$$

The *iid* assumption simplifies the analysis since it enables us to compare the autarky to the under-contract values by looking only one period ahead. In fact,

$$v_a(\delta_1) - v_c(\delta_1, \mu) = u(y_1) - u(c_1(\mu)) + \beta \left( \mu_1 v_a(\delta_1) + \mu_2 v_a(\delta_2) - \mu_1 v_c(\delta_1, \mu) - \mu_2 v_c(\delta_2, \mu) \right),$$

which, after noting that the autarky and contract values coincide for the participation-constrained state and that next-period's under-contract value depends on the same probability as the contemporary period's (this is the *iid* assumption), yields

$$v_a(\delta_1) - v_c(\delta_1, \mu) = \frac{u(y_1) - u(c_1)}{1 - \beta\mu_1}$$

We can use this information and plug it into (4.3) to obtain the optimal reaction of the type-1 agent (facing every type-2 agent running) to be to withdraw if

$$\frac{y_1 u(c_1) - u(y_1) c_1}{y_1 u(c_1) - u(y_1) y_1} \leq -\frac{\beta\mu_1}{1 - \beta\mu_1}.$$

Whether this inequality is satisfied depends on the utility function and the income process, namely, on the degree of risk-aversion and the amount of risk. If it is, there are no partial runs: when people expect the bank to fail, everyone wants out of the contract. On the other hand, if the inequality is not satisfied, the type-1 agents are induced to stay in the contract even though know that the bank is failing. This is because, even though they realize that they might lose their current consumption, they still value the possibility of staying in the contract in the future, when the run is over.

Furthermore, since we are assuming that the income process is *iid*, then next period's measure of agents is the original $\mu$, so the contract is exactly replicated –at scale $\mu_1$.

## 4.1 Runs under *iid* shocks

We have shown that partial runs arise naturally in our model. What we did not address is the issue of what happens after a partial run. In the previous section, we assumed that an agent ahead in line -and who managed to get paid- could continue in the contract, so the run was confined to that first period. We now turn to the case of runs that take time.

[11] which implies that $v_c(\delta_2, \mu) = v_a(\delta_2)$ for every probability $\mu$

Maintaining the *iid* assumption, assume in general that it is common knowledge that type-$S$ agents will choose to withdraw every period. As before, the bind on their participation constraint will imply that –at least– this strategy is compatible with equilibrium. In the last section, we showed that, in the two-state case, type-1 agents might find it optimal to stay in the contract even when a run is under way, and that this implied that the contract was unaffected except for its scale. However,

**Lemma 4.2.** *There are no dynamic runs in the two-state case.*

This Lemma shows that, in the two-state case, when high-income agents are expected to deviate from the contract in the future as well as in the present, it is pointless for a low-income agent to remain. Really, this happens because of a technical reason: with only two states, if agents in one of those states withdraw, then the remaining agents can extract no insurance from the contract. In terms of Section 3, when all agents in states of class 2 are withdrawing, the only reason why agents in states of class 1 might choose to remain is to obtain insurance *between states of class 1*.

We will show that with three states, two of which are low-income states (in the sense that the participation constraint does not bind[12]), dynamic runs can arise. In these runs, the high-income type-3 agents will withdraw, while the low-income type-1 and type-2 agents will choose to remain in the contract. The *iid* assumption allows us to start from the following result.

**Lemma 4.3.** *In the iid case, the only relevant strategies are to withdraw whenever a state higher than $s$ is reached.*

In a dynamic run, the optimal strategy for all agents will be to wait until they reach type-3, and revert to autarky at that point. Let then $T_1$ be the (random) time when this happens. That is, let $T_1 = \min\{t \geq 0 : y(t) = y_3\}$. Because of the *iid* assumption, $T_1$ has a geometric distribution with parameter $\mu_3$.

Now, if every agent chooses this strategy, the bank will be insolvent every period, so a fraction of agents will be too late in line. These agents will miss the contemporary payment and will forcefully revert to autarky. Let $T_2$ be the random time at which an agent is late in line. Since the time-$t$ place in line is perceived as following a uniform distribution, $T_2$ is distributed geometrically with parameter $1 - \frac{\mu_1 y_1 + \mu_2 y_2}{(\mu_1+\mu_2)c_1}$.

Finally, let $T = \min(T_1, T_2)$. This is a stopping time with respect to time-$t$ information, $\mathcal{F}_t$. This is the time at which an agent gets out of the contract, either because of choice (because state 3 has been reached and the agent's strategy is to withdraw) or because she has been forced to (because she was late in line).

We now turn to the characterization of the dynamic run. Because of Lemma 4.3, we only need to show that the strategy of remaining in the contract until $T_1$ (ie, until the type-3 shock) is preferred to the strategy of withdrawing, under states 1 and 2. Moreover, since in state 2, the deviation to autarky is strictly more tempting than in state 1, we only need to check that the value of remaining in the contract until reaching state 3 is preferred to autarky starting from state 2.

Denote the value of staying in the contract until $T$ arrives, when state is $s$, by $v_e(\delta_s)$. Then, $v_e(\delta_s)$ sums the utility of imperfectly-insured consumption, over time ranging from zero to $T$, and then adds the autarky value from then on, with the following caveat. If the agent was forced out the contract because she was late in line (that is, if $T = T_2$), her consumption will be zero in that period, and autarky in the future. On the other hand, if an agent withdraws because state 3 has arrived, her current consumption will be $y_3$. Then,

$$v_e(\delta_s) = \mathbb{E}\left[\sum_{t=0}^{T-1} u(c_1)\beta^t + \mathbf{1}_{\{T=T_1\}}\beta^T v_a(\delta_3) + \mathbf{1}_{\{T=T_2\}}\beta^{T+1} v_a(\delta_{y(T)}P) | y(0) = y_s\right].$$

[12]as before, this will imply $c_1 = c_2$

Notice that, since the decision of staying or leaving the bank has no impact on income (because the latter is exogenous), the autarky terms do not depend on any decisions (in particular, they are independent of the time of withdrawal). Then, when an agent is considering withdrawing, the only concern to worry about is the possibility of not getting any consumption at time $T_2$, provided it arrives before time $T_1$. So, we have that

$$v_e(\delta_s) - v_a(\delta_s) = \mathbb{E}\left[\sum_{t=0}^{T-1} \beta^t \left(u(c_1) - u(y(t))\right) - \mathbf{1}_{\{T=T_2\}} \beta^T u(y(T)) | y(0) = y_s\right] \tag{4.4}$$

**Theorem 4.4.** *In the iid three-state case, there will be a dynamic run, meaning that all agents choose to wait until $y(t) = y_3$ to withdraw, if the following condition is satisfied*

$$(u(c_1) - u(y_2)) \frac{\mu_1 y_1 + \mu_2 y_2}{(\mu_1 + \mu_2) c_1} + \left(u(c_1) - \frac{\mu_1 u(y_1) + \mu_2 u(y_2)}{\mu_1 + \mu_2}\right) \frac{\beta\left(\frac{\mu_1 y_1 + \mu_2 y_2}{c_1}\right)}{1 - \beta \frac{\mu_1 y_1 + \mu_2 y_2}{c_1}} \frac{\mu_1 y_1 + \mu_2 y_2}{(\mu_1 + \mu_2) c_1} > \\ > \mathbb{E}\left[\mathbf{1}_{\{T=T_2\}} u(y(T)) \beta^T | y(0) = y_2\right], \tag{4.5}$$

*where the expectation on the right-hand side equals*

$$u(y_2)\mathbb{P}(T_2 = 0) + \sum_{t=1}^{\infty} \beta^t \frac{\mu_1 u(y_1) + \mu_2 u(y_2)}{\mu_1 + \mu_2} \mathbb{P}(T_2 = t)$$

*and the probabilities concerning $T_2$ are computed by using the geometric distribution.*

*Otherwise, if the condition is not satisfied, the run will be massive and synchronized: everyone will deviate at the same time.*

This proposition simply observes that a type-2 agent has to trade off the benefit of smoothing consumption between states 1 and 2, weighted by how long they can expect to be able to do so, against the possibility of being late in line on one particular period and having no consumption at that time. The condition for type-2 agents to stay in the contract is complicated because it involves a difficult decision, which consists in buying risk-sharing –for some random time– with taking up some other risk –at another random time.

Theorem 4.4 shows that the strategy of waiting until state 3 to withdraw can be an equilibrium. In this equilibrium, there is a perpetual run: each period, a fraction $\mu_3$ of agents withdraws, the bank fails to keep a fraction $1 - \frac{\mu_1 y_1 + \mu_2 y_2}{(\mu_1 + \mu_2) c_1}$ of its remaining liabilities, and the same fraction of agents that are still in the bank (uniformly distributed between states 1 and 2) gets no consumption and is forced into autarky, while the rest of the agents in contract receive their payments and stay in the bank in the following period.

## 4.2 Markov shocks

Now that we have exploited some intuitions of the *iid* case, we will attempt to translate these ideas to the original setting with a Markovian income stream. Throughout this section, we will keep the three-state setup, and the assumption of common knowledge on the fact that every type-3 agent is withdrawing. Notice that this last assumption is, as noted before, an essential part of equilibrium: in the event of a run, the bank is failing at least in some states of nature at some point in the future, and so the participation-constrained type-3 agents need to withdraw immediately.

We will make two simplifying assumptions. First, that we start from the invariant measure for the income process (that is, $\mu P = \mu$). This implies that the promised consumptions $c_s$ are constant over time. However, during a dynamic run, the type-3 agents will withdraw. Since withdrawing agents cannot go back into the contract, the distribution of states for agents under the contract will not accurately represent the actual distribution of states. Therefore, the distribution of states under the contract will not coincide with the invariant distribution $\mu$ for states 1 and 2, and will therefore have its own dynamic.

Second, we will impose the technical requirement that $\frac{p_{22}}{1-p_{23}} \geq \frac{p_{12}}{1-p_{13}}$ This condition states that, conditional on not going to state 3, the probability of remaining in state 2 is greater than the probability of going from state 1 to state 2. This is meant to ensure that state 2 is preferred to state 1, and that the agent demands insurance against going into state 1. Also, we will maintain the assumption that the utility function and the process are such that the participation constraint is never binding for type-2 agents, so that $c_2 = c_1$.

In this section, we are interpreting $\mu$ as the distribution of endowment *as perceived by the bank*, that is, among those agents that remain in the contract at the beginning of the period. This distinction was irrelevant in the *iid* case, because the distribution of endowment was independent of decisions made in the past, while it is now affected by how many agents were in each state last period. Now, $\tilde{\mu}$ is the distribution of income for the agents that were never in state 3 (and were not defaulted on by the bank). Thus, $\tilde{\mu}_s(t) = \mathbb{P}\left(y(t) = y_s \mid y(l) \neq y_3, \forall l \leq t\right) = \mathbb{P}\left(y(t) = y_s \mid y(t-1) \neq y_3 \neq y(t)\right)$. We can therefore write

$$\tilde{\mu}'(\nu) = \left(\nu_1 \frac{p_{11}}{1-p_{13}} + \nu_2 \frac{p_{21}}{1-p_{23}}, \nu_1 \frac{p_{12}}{1-p_{13}} + \nu_2 \frac{p_{22}}{1-p_{23}}\right) \tag{4.6}$$

Let $v_e(\lambda, \tilde{\mu})$ be the expected value of an agent choosing the optimal withdrawing policy, with a random state chosen with the probability $\lambda$, when the distribution of endowment among agents in the contract is $\tilde{\mu}$, and facing every type-3 agent running every period.[13] As opposed to the *iid* case, generally we will have $\frac{1}{\mu_1+\mu_2}(\mu_1, \mu_2) \neq \tilde{\mu}$. This is due to the fact that, when we allow persistence on the income shock, conditioning on the fact that the agent has never reached state 3 in the past ceases to be innocuous. However, since we have a Markovian structure, the distribution among agents that never reached state 3 coincides with the (current) distribution among agents that were not in state 3 in the last period. Then,

$$v_e(\delta_s, \tilde{\mu}) = \begin{cases} v_a(\delta_3) & \text{if s = 3} \\ \max\left\{v_a(\delta_s), \frac{\tilde{\mu}_1 y_1 + \tilde{\mu}_2 y_2}{c_1}\left(u(c_1) + \beta v_e(\delta_s P, \tilde{\mu}')\right) + \left(1 - \frac{\tilde{\mu}_1 y_1 + \tilde{\mu}_2 y_2}{c_1}\right)\beta v_a(\delta_s P)\right\} & \text{otherwise} \end{cases}$$

Thus, the agent in state $s$ faces an incentive to stay in the contract today given by

$$v_e(\delta_s, \tilde{\mu}) - v_a(\delta_s) = \max\left\{0, \frac{\tilde{\mu}_1 y_1 + \tilde{\mu}_2 y_2}{c_1}\left(u(c_1) + \beta(v_e(\delta_s P, \tilde{\mu}') - v_a(\delta_s P))\right) - u(y_s)\right\} \tag{4.7}$$

The Bellman equation (4.7) depicts the same problem we already faced in the last section. The agent decides when to withdraw, knowing that staying in the contract delivers $u(c_1)$ and the continuation value of choosing again next period, with a probability essentially given by the bank's balance sheet, but that with the rest of the probability it leaves the agent in autarky forever and with no consumption today. For this equation to tell us that there is an equilibrium in which everyone waits until state 3 to withdraw, we need to show that there are combinations of $s$ and $\tilde{\mu}$ for which $v_e(\delta_s, \tilde{\mu}) > v_a(\delta_s)$. To do so, first notice that $v_e(\delta_s, \tilde{\mu}) - v_a(\delta_s)$ is implicitly defined as a fixed point of the functional operator in (4.7), interpreting, for each $s$, (4.7) as a mapping between functions of $(\tilde{\mu}_1, \tilde{\mu}_2)$.

Notice that remaining in the contract is more appealing $\tilde{\mu}_2$ is greater, because the bank has more

[13] notice that $\tilde{\mu}$ is a probability between states 1 and 2

resources. Also, notice that the restriction we imposed, that $\frac{p_{22}}{1-p_{23}} \geq \frac{p_{12}}{1-p_{13}}$, implies that next-period's $\tilde{\mu}_2$ will be increasing in current $\tilde{\mu}_2$, as can be seen in (4.6). Both these facts will prove helpful in proving the following result.

**Lemma 4.5.** *The operator in (4.7) maps nonnegative differentiable increasing functions onto nonnegative differentiable increasing functions.*

Since it is straightforward that the operator in equation (4.7) satisfies the Blackwell sufficient conditions, then by Theorem 9.7 (from Stokey and Lucas, 1989), and the previous Lemma 4.5, we can conclude that it is a contraction on the space of continuous non-decreasing nonnegative functions, and must therefore exhibit a fixed point in that space. All that remains is to show that the value function $v_e(\delta_s, \tilde{\mu}) - v_a(\delta_s)$ is strictly positive at some point. To show it, we will put ourselves in the most appealing scenario to be on the contract, namely, the case in which the aggregate state is $\tilde{\mu}_1 = 0$, whereas the agent is on state 1.[14] In this scenario, the bank's resources are maximized and we have that $v_e(\delta_1, (0,1)) > v_a(\delta_1)$ if and only if

$$y_2 u(c_1) + \beta(1 - p_{13}) \left[ v_e \left( \tilde{\mu}'(0,1), \tilde{\mu}'(1,0) \right) - v_a \left( \tilde{\mu}'(1,0) \right) \right] > c_1 u(y_1)$$

Since $v_e(\nu, \lambda) - v_a(\nu) \geq 0$ by definition, a sufficient condition for the type-1 agent to be willing to stay in the contract is that

$$\frac{u(c_1)}{u(y_1)} > \frac{c_1}{y_1} \frac{y_1}{y_2}. \tag{4.8}$$

Condition (4.8) is automatically satisfied by every risk-neutral or risk-averse utility function. Indeed, for every (weakly) concave $u$, $\frac{u(c_1)}{u(y_1)} \geq \frac{c_1}{y_1}$ holds. Further, $\frac{y_1}{y_2} < 1$ is true by definition. A similar condition arises when the agent is assumed to be in state 2, namely, that $v_e(\delta_2, (0,1)) > v_a(\delta_2)$ if $\frac{u(c_1)}{u(y_2)} > \frac{c_1}{y_2}$, which is satisfied if $u$ is assumed to be concave.

Putting together these two pieces, that is, Lemma 4.5 and the observation that $v_e - v_a$ is strictly positive in some region of its domain, we obtain the following

**Theorem 4.6.** *There is a threshold $\mu_2^\star$ on $\tilde{\mu}_2 = \mathbb{P}(y(t) = y_2 \mid y(t-1) \neq y_3 \neq y(t))$ such that if $\tilde{\mu}_2 \geq \mu_2^\star$, then every agent adopts the strategy of waiting until they are type-3 to withdraw from the bank.*

Theorem (4.6) shows that the agents are more willing to stay in the contract -when there is a run going on- when they believe the bank's resources to be enough to cover a larger part of its liabilities –that is, when $\tilde{\mu}_2$ is large relative to $\tilde{\mu}_1$.

# 5 Concluding remarks

In this paper, I develop a dynamic model of bank runs. I show that, under certain conditions, an agent's optimal reaction to an ongoing run might be to remain in the contract now while planning to withdraw in the future. This feature can help accomodate actual episodes in which depositors withdrew steadily over a long time. A modified interpretation could also accomodate episodes of runs that started with a slow decline in deposits followed by an abrupt acceleration at a certain point.

Theorem 4.6 shows that there is a critical value $\tilde{\mu}_2^\star$ so that, if the fraction of agents in state 2 is less than $\tilde{\mu}_2^\star$, all agents will run together, while an agent will wait until state 3 to withdraw if that fraction is greater than $\tilde{\mu}_2^\star$. While we assumed that $\mu$ was the invariant measure for the income process, $\tilde{\mu}$ was

[14] this situation can be thought of either by understanding that a single agent has measure zero and thus can be in a state whose measure under the aggregate distribution is zero, or by considering a limit case where $\mu_1 \to 0$ while the individual state is always 1

not invariant. Therefore, it is conceivable that a run might have started with $\tilde{\mu}_2$ above the threshold and the dynamic of the run made $\tilde{\mu}_2$ cross it at some point. Although artificial, this would be a way to accomodate the slow decline followed by sharp acceleration pattern of withdrawals. An improvement upon it is suggested in Section 5.1.

The occurrence of partial runs in this model has some regulation implications, as I show that these are intimately related to what ultimately happens to a bank that has failed to repay some of its obligations. The occurrence of partial runs is also strongly associated to the solidness of the bank's balance sheet (through the critical amount $\tilde{\mu}_2$). Also, this model stresses that, in order to stop a run with bailouts or suspension of convertibility, the government should be willing to implement the policy forever. If it were not disposed to do so -because of, among others, moral hazard considerations-, then it should implement another active policy to change the public's expectations. This is in line with the findings of Ennis and Keister (2009), who show that a government that cannot pre-commit to follow through with a suspension-of-convertibility threat will not be able to stop runs.

Finally, I have shown that liquidation costs are not essential to the issue of bank runs, but rather a necessity imposed by the taste-shock microfoundation usually chosen. When the demand for insurance comes from randomness of the income stream, the bank can exploit other trade opportunities to provide insurance to its depositors. Also, this new microfoundation implies that the class of dynamic contracts with limited commitment is subject to runs. The reason behind this result is that the incentive-compatibility constraint presupposes that the rest of the agents are not withdrawing; however, this will obviously not be true when a run is under way.

## 5.1 Further Research

A possible extension of this work would consist in removing the perishable good assumption. There are two main reasons why this model would benefit from the availability of a storage technology. First, it would be necessary (by consistency) in order to allow for investment and liquidation costs on the bank's part. Several new strategies would then be available to the agents, including the strategy they followed in the DD model: withdrawing even if they wanted to consume later.

Second, a storage technology would allow the agents to insure themselves, as in Chamberlain and Wilson (2000). However, self-insurance would not implement the first-best allocation.[15] Moreover, individual assets would converge to infinity in a stochastic income environment, while the bank faces a deterministic environment. However, even without aggregate uncertainty, the bank's problem would not be a deterministic self-insurance problem. Indeed, current assets will enter the bank's objective function through the agents' participation constraints. On the one hand, a bank with more assets can (and, in equilibrium, will) promise more consumption. On the other hand, a higher face value of deposits rises the outside option for every agent.

Another possible extension concerns the stationary distribution assumption in Section 4.2. There is one main reason to lift this technical assumption, to allow for a "macroeconomic" trend in income. If the result in Theorem 4.6 could be extended to that case, then the result from Krugman (1979) could also be extended to this setup in a less artificial way: as general income diminishes (during the recession), only the agents with the highest shocks withdraw. However, when a certain threshold is attained, it ceases to be optimal to wait for a favorable shock to exit, and there is a massive run.

In this model, the bank wields an unrealistic amount of power. Really, the assumption that the bank can commit to punish a deviating agent with autarky forever is perhaps better suited for other contracts. It is certainly best suited for cases in which the threat is implicit and never implemented in equilibrium. However, in a model of runs, the threat will be executed. Therefore, another possible extension regards allowing deviating agents to return into the contract, perhaps after some time has

[15] see Karaivanov and Martin (2011)

passed. Choosing this time optimally is likely to be a problem of its own right. In any case, this extension would rise the outside option and it should, if anything, make runs more likely.

The last extension concerns the assumption that the banks behave in perfect competition. While I assume the bank to be competitive, the bank's punishment ability hinges on the fact that there is only one bank. There are two ways to solve this problem. The first one consists in assuming that the banks are in fact well-communicated and therefore know if a given depositor has defected a contract in the past. The second solution –to assume that depositors can switch banks– would be equivalent to a microfoundation for the extension considered above, in which banks can only threaten with autarky for a finite length of time. Therefore, these last consideration would, if anything, make runs more likely.

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# A Appendix

**Proof of Lemma 3.1** There are only two autarky values, and we obtain by substracting

$$v_a(\delta_2) - v_a(\delta_1) = \frac{u(y_2) - u(y_1)}{1 + \beta(p_{21} - p_{11})}.$$

Therefore, since $p_{21} - p_{11} \geq -1 > -1/\beta$, we have $v_a(\delta_2) > v_a(\delta_1)$. Notice that, in this small state-space, the agent always prefers to go into autarky with a high income: there just are not enough states for transition probabilities to shift the continuation values in order to offset the initial difference. Thus, the constraint on the low state cannot be the only binding one.

Now, recall the bank's problem first-order condition $u'(c_s)\frac{\mu_s + \lambda_s}{\mu_s} = \eta \geq 0$. From this equation, we know that if no constraint binds, then $\lambda_s = 0$ and so $c_1 = c_2$. If only the constraint on the high state binds, we have $\lambda_2 > \lambda_1 = 0$, which implies $u'(c_1) > u'(c_2)$ and $c_2 > c_1$.

If both constraints bind, it must be that $c_1 < y_1$. Therefore, using that non-storability implies full usage of resources, we have $c_2 \geq y_2 > y_1 \geq c_1$. Strict inequalities will be absurd since the contract would then leave agents worse off *ex-ante*. On the other hand, equalities would imply that the process and the utility function are such that the contract cannot improve the allocation. We will abstract from this case but yet point out that it is trivial since no gains are available. □

**Proof of Proposition 3.2** Assume that only $\alpha$ of the type-2 agents stay in the contract. Then, if the bank fails, every type-2 agent will want to withdraw. In this case, the type-1 agents have to strategies: to withdraw and earn $y_1$ with certainty, or to stay and earn $c_1$ with a probability such that the expected payoff is $y_1$ (if all type-2 agents withdraw, the bank only has the deposits of the type-1 agents). Any risk-averse type-1 agent will thus withdraw.

If, on the other hand, the bank does not fail, it must be the case that some type-1 agents have withdrawn, since the bank fails when only type-2 agents withdraw. But if the bank holds, withdrawing is suboptimal for the type-1 agents. Therefore, the only equilibrium run in the two-state model is a massive run. □

**Proof of Theorem 3.3** The proof consists of showing that partial runs cannot occur in equilibrium. Assume that there is a run, and that a positive fraction of agents withdraw, while a positive fraction of agents stay. There are two possibilities. When there is a run, the bank may fail or hold. We will explore expectations that lead to both kinds of runs and analyse the agents' reactions to those expectations and show that all lead to contradictions.

If the agents' expectations imply that the bank will hold, agents in states of class 1, whose participation constraint is not binding, will be strictly better off remaining in the contract. In turn, agents in states of class 2 are indifferent between staying or leaving. If they all stay, the existence of the run is contradicted. On the other hand, if some of them withdraw, the assumption that the bank holds is contradicted.

If the agents' expectations imply that the bank will fail, agents in states of class 2, whose participation constraint binds under the optimal contract, now find it optimal to withdraw. Therefore, the bank's resources consist, at best, of the income from all agents in states of class 1. Now, if the bank is failing, choosing whether to withdraw or to run comes down to comparing contemporaneous outcomes. Since the bank is failing, the agent will be in autarky from next period onwards regardless of her decisions today.

Now, consider the choice of those agents in the state of class 1 with the highest contemporaneous income. These agents can withdraw and take their current income with certainty, or stay and get $c^{(1)}$ with a probability that makes the expected payoff equal to the bank's resources. Expected payoff equals the bank's resources because, when every agent in states of class 2 has withdrawn, the bank is committed to paying everyone left in the contract the amount $c^{(1)}$. Now, the bank's resources consist of the income from agents in states of class 1. Therefore, the expected payoff of staying in the contract cannot exceed the contemporaneous income of the agent in the state of class 1 with the highest contemporaneous income. These agents will therefore withdraw. Repeating this argument leads to conclude that, if agents are expecting a run that will make the bank fail, all agents will choose to withdraw.

Therefore, only the massive run can arise in equilibrium. $\square$

**Proof of Lemma 4.1** Assume there is a run. If the bank is failing, every type-2 agent needs to withdraw since their participation constraint is binding when the bank is solid.

We need to show that the bank will fail under any run. Assume to the contrary that a given run implies that the bank will be solid. No agent of type-1 will participate of such a run, since they are strictly better off under the contract (and the bank's solidness implies that payment occurs with probability one). Now, if there is a run and the type-1 agents are not participating, it must be that some type-2 agents are running. This contradicts the bank's solidness: if only type-2 agents are running, the bank must fail. $\square$

**Proof of Lemma 4.2** Notice that with only two states, the only dynamic run consists of everyone waiting until state 2 to withdraw. But this strategy is pointless for the type-1 agent: every period, she is getting $c_1$ with a probability that implies that the expected payoff is $y_1$. Autarky is a much better option: as long as state 2 is not reached, her (certain) payoff equals the expected payoff of remaining in the contract, and both strategies yield the same payoff when state 2 is reached.

Therefore, when expecting all agents to follow the strategy of waiting until state 2 arrives to withdraw, every agent's reaction consists of withdrawing instantly, which leads to a massive run. $\square$

**Proof of Lemma 4.3** In the *iid* setup the natural ordering of autarky values applies: with the same continuation value, nothing can offset a better income today. Therefore, any strategy that implies withdrawing in state $l$ and remaining in state $k$, with $y_k > y_l$, is suboptimal.

Now, since the setup of the model is ergodic, it cannot be optimal to remain when the state is $s$ and to withdraw in the future in state $s$. In fact, the only state variable is the distribution $\mu(t)$, which is a constant under the *iid* assumption. Therefore, every optimal strategy will have the form of remaining while the state is lower than some threshold state $s \in \{0, 1, ..., S\}$. $\square$

**Proof of Theorem 4.4** Because of Lemma 4.3, we only need to consider the class of strategies that imply waiting until state $s$ is reached to withdraw.

The exact condition follows with some algebra about random times. Start with

$$\begin{aligned} v_e(\delta_2) - v_a(\delta_2) &= \mathbb{E}\left[\sum_{t=0}^{T-1} \beta^t (u(c_1) - u(y(t))) - \mathbf{1}_{\{T=T_2\}} \beta^{T+1} u(y(T+1)) | y(0) = y_2\right] \\ &= \left(\sum_{t=0}^{\infty} \mathbb{E}\left[\mathbf{1}_{\{t<T\}} \beta^t (u(c_1) - u(y(t))) | y(0) = y_2\right]\right) - \mathbb{E}\left[\mathbf{1}_{\{T=T_2\}} \beta^T u(y(T)) | y(0) = y_2\right] \end{aligned}$$

Now use iterated expectations by conditioning on $T$, the conditional distribution of $y(t)$ given $t < T$, the distribution of $T$, and that $y(0) = y_2$, to get the first term to become

$$(u(c_1) - u(y_2))\mathbb{P}(T > 0) + \left(u(c_1) - \frac{\mu_1 u(y_1) + \mu_2 u(y_2)}{\mu_1 + \mu_2}\right) \sum_{t=1}^{\infty} \beta^t \mathbb{P}(T > t)$$

Finally, observe that $\mathbb{P}(T > t) = (1 - \mu_3)^t \left(\frac{\mu_1 y_1 + \mu_2 y_2}{(\mu_1 + \mu_2) c_1}\right)^{t+1} = \left(\frac{\mu_1 y_1 + \mu_2 y_2}{c_1}\right)^{t+1}$, and use the geometric series to get the first term.

For the second, use total probability to express the expectation in terms of the expectation conditional on $T = T_2$ and $T \neq T_2$ –and observe that only the first part survives. Then use that, because of the *iid* assumption, the distribution of $y(T_2)$ is degenerated at $y_s$ if $T_2 = 0$ and just the re-normalized $\mu$ between $y_1$ and $y_2$ otherwise.

Finally, notice that, if $v_e(\delta_2) < v_a(\delta_2)$, then everyone on state 2 is also withdrawing. But then, the type-1 agents prefer to withdraw too: taking the strategy of waiting until leaving state 1 to withdraw yields the same expected payment than autarky but with a greater variance. Indeed, each period they get the type-1 shock, consumption is $c_1$ conditional on getting paid, but the expected consumption is $\frac{y_1}{c_1}c_1$, whereas they can get $y_1$ for sure by withdrawing now. Notice that, because of Lemma 4.3, we only need to consider this kind of "static" strategies, so the proof is complete. □

**Proof of Lemma 4.5** First of all, observe that, since the bank only has agents of types 1 and 2, we can use $x = \tilde{\mu}_2$ as a state variable in equation (4.7) to rewrite it as

$$f(s,x) = \max\left\{0, \frac{(1-x)y_1 + xy_2}{c_1}\left[u(c_1) + \beta\left(p_{s1}f\left(1,x'\right) + p_{s2}f\left(2,x'\right)\right)\right] - u(y_s)\right\} = Tf(s,x).$$

where, making use of (4.6), $x' = (1-x)\frac{p_{12}}{1-p_{13}} + x\frac{p_{22}}{1-p_{23}}$.

Notice that $\frac{\partial x'}{\partial x} = \frac{p_{22}}{1-p_{23}} - \frac{p_{12}}{1-p_{13}}$. Now let $f$ be a nonnegative differentiable (on its second argument) increasing function, and let $f_1 = Tf$. Then, we have

$$\begin{aligned}\frac{\partial f_1}{\partial x}(s,x) = \quad & \frac{y_2 - y_1}{c_1}\left[u(c_1) + \beta\left(p_{s1}f(1,x') + p_{s2}f(2,x')\right)\right] + ...\\ & + \frac{(1-x)y_1 + xy_2}{c_1}\left[\beta\left(p_{s1}\frac{\partial f}{\partial x}(1,x') + p_{s2}\frac{\partial f}{\partial x}(2,x')\right)\left(\frac{p_{22}}{1-p_{23}} - \frac{p_{12}}{1-p_{13}}\right)\right]\end{aligned}$$

Since $y_2 > y_1$ and $f$ was assumed to be nonnegative, the first term is positive. Therefore, for the result to hold, this equation implies a condition on the transition probabilities, namely, that the -normalized- probability of remaining in state 2 cannot be excessively smaller than the probability of entering state 2 from state 1. This condition, which we imposed on the income process, means that the spell of state 1 should be large compared to the spell of state 2, so that the event of entering state 1 is bad enough for an insurance to be demanded –even at the cost of missing one payment at some point.

Notice that in the *iid* case the second term is always zero -because $\frac{\partial \tilde{\mu}'}{\partial \tilde{\mu}} = 0$. Hence, this operator maps any function into an increasing one in the *iid* case. □

**Proof of Theorem 4.6** Take the Bellman equation (4.7), and notice that the operator is a maximum between something that is, as we have shown, increasing in $\tilde{\mu}_2$, and zero. Further, we know that at some point, namely when $\tilde{\mu}_2 = 1$, the incentive to stay in the contract (on states 1 or 2) is strictly positive. This value function is also continuous, since it is defined as a (uniform) limit of differentiable functions. Therefore, there must exist a threshold $\mu_2^\star$ such that the agent is indifferent between staying in the contract and waiting for state 3 to withdraw. This is the desired threshold. □