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**“Growing Directed Networks: Limit in-degree
distribution for arbitrary out-degree one”**

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We compute the limiting in-degree distribution for a growing network process with directed edges and arbitrary out-degree distribution. In particular, under preferential linking, we find that if a process is such that each new node has a light tail (finite variance) out-degree distribution, then the corresponding in-degree one behaves as k^{-3} . Moreover, for an out-degree distribution with a scale invariant tail, $P_{out}(k) \sim k^{-\alpha}$, the corresponding in-degree distribution has exactly the same asymptotic behavior only if $2 < \alpha < 3$ (infinite variance). Similar results are obtained when attractiveness is included. The results presented here explain the in/out-degree distribution observed in many real networks.

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The study of networks has attracted many scientists during the last decade, specially after it was discovered that empirical growing networks have a scale-free stationary “state” [1–3]. This state is characterized, among other things, by the fact that it has a degree distribution with a power law tail, where the degree of a vertex is defined as the total number of its connections. In the case of directed networks, the in-degree is defined as the number of incoming edges of a vertex, and the out-degree as the number of its outgoing edges. The most studied directed growing networks are the WWW network [4], where each node represents a web page and the hyper-links (references to other web pages) represents the directed edges or links, and the scientific papers network [5], where each paper is a node, and its references the directed links.

Empirical directed growing networks follow in general one of two possible behaviors. In the first case they have an out-degree exponential distribution, $P_{out}(k) \sim a^k$ ($0 < a < 1$), or an out-degree distribution taking finitely many values, associated with an in-degree one distribution with a power law tail $P_{in}(k) \sim k^{-\alpha}$ where typically $\alpha \approx 3$. In the second case the out-degree distribution satisfies $P_{out}(k) \sim k^{-\beta}$, and is associated with $P_{in}(k) \sim k^{-\alpha}$ with $\alpha \approx \beta$. Examples, such as biological, WWW, or communication networks, can be found in [1–3].

In this letter, we address the question of why the empirical growing directed networks show this strange general behavior for the in/out degree distributions. We study a particular growing network model (a generalization of [6] to be precise), obtaining the limiting joint in-out degree distribution, and some of its derivatives, such as the marginal distribution and covariance. In particular we probe that, at least for the model we consider, it is expect to observe the in/out degree distribution reported for real networks [1–3].

Let us now describe the growing network process (Fig. 1), GNP: as time evolves new nodes with D_{out} number of out (directed) links appear, which connect to the existing nodes according to some probability law (uniform,

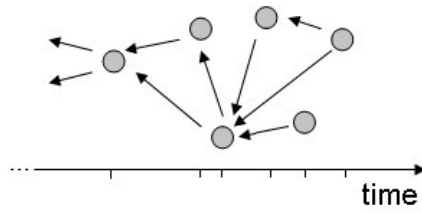


FIG. 1: Scheme of the growing network model. As time evolves new nodes with D_{out} number of out-links appear, these ones attaches to the existing nodes.

preferential linking, etc.). This model is in fact an extension of the Albert-Barabási model, now D_{out} is a random variable with an arbitrary distribution, $P(D_{out} = j) = p_j$ with $j \in N$. Fig. 1 shows an example where $D_{out} = \{1, 2\}$ ($P(D_{out} = 2) = p_2, P(D_{out} = 1) = 1 - p_2$). Note that even in this extremely simple case, the limit in-degree distribution is unknown, and the same applies to the total degree distribution. Moreover, it is not known if the limit distributions satisfy a superposition principle (linear combination).

It is generally accepted that a non-homogeneous Poisson process provides a good way of describing the arrival of the new nodes. In this paper we are interested in limit (asymptotic) distributions, though, which are independent of the arrival process except in the pathological cases where explosions might occur, $P(\text{number of nodes}(t) = \infty) > 0$. It is therefore enough to study the time step process, where in each step a new node with D_{out} out links is aggregated following the previous rule. Clearly, as the out-degree does not depend on time, the limit out-degree distribution satisfies $\nu_{out}^k \equiv \lim_{t \rightarrow \infty} P(\text{out-degree}(t) = k) = p_k$. We are interested in obtaining the limit degree distribution ν_{deg}^k , the in-degree one ν_{in}^k , and the joint distribution $\nu_{in,out}^{j,k}$.

Initially, the network consists of m nodes connected in a given way. Let $\vec{X}_i = (X_i^1, X_i^2, \dots, X_i^n, \dots)$ denote the degree vector, with k -th component standing for the number of degree k nodes present at time step i ($\vec{X}_i \in N^{N_0}$ with $i \in N_0 \equiv N \cup \{0\}$). From now on, supraindexes will denote degree number while subindexes indicate time steps. At each time step, say time step $n + 1$, a node with D_{out} links appear, and each new (directed) link is connected to an existing node with probability π_{n+1} . If π_{n+1} is an arbitrary function that depends only on \vec{X}_n and/or \vec{X}_n^{in} (\vec{X}_n^{out}), then the GNP described above is a Markov chain taking values in N_0^N or $N_0 \times N_0^{N^2}$ with transition probabilities given by π_{n+1} .

We will associate to the transition probabilities of this Markov chain some random variables that we now describe. In the first place, there is the out-degree, D_{out} , of the new node. Secondly, we consider at each time $n + 1$ a sequence of i.i.d. random variables $\{Y_i\}_{1 \leq k \leq n}$, taking value k , $k \in N$, with probability π_{n+1}^k , which depends on the state of the chain at time n . This way, the GNP dynamics is:

$$X_{n+1}^k = X_n^k + \delta_{D_{out}=k} + \sum_{i=1}^{D_{out}} \delta_{Y_i=k-1} - \delta_{Y_i=k}, \quad \forall k \in N. \quad (1)$$

If at time $n + 1$ a new node with $D_{out} = m$ out-links is aggregated, then the m -th degree vector component grows by

one, and D_{out} components of the degree vector undergo a “shifts” of the kind $\vec{X}_{n+1} = (\dots, X_n^k - 1, X_n^{k+1} + 1, X_n^{k+2}, \dots)$. As the network continues to grow, the goal is to find whether there exists a limit distribution for the (in) degree. For very large values of n , given a randomly selected node, what is the probability that it has k links (or j/k in/out-links)? Let us denote by $D \in N$ the degree of the randomly chosen node. We want to determine:

$$\nu_{deg}^k := P(D = k) = \lim_{n \rightarrow \infty} \frac{X_n^k}{\sum_{k \in N} X_n^k}, \quad (2)$$

where the last equality holds by the Law of Large Numbers. The following property shows a new way of computing $\vec{\nu} = (\nu_{deg}^1, \nu_{deg}^2, \dots)$, which has interest on itself.

Property 1: $\vec{\nu}$ is the solution of:

$$E\left(\frac{X_{n+1}^k}{\sum_{k \in N} X_{n+1}^k} / \frac{\vec{X}_n}{\sum_{k \in N} X_n^k} = \vec{\nu}\right) = \nu_{deg}^k \quad \forall k \in N. \quad (3)$$

The preceding property says that if the process is distributed according to the limit measure, $\vec{\nu}$, it will stay there in expectation. Note that the $\vec{Z}_n \equiv \frac{\vec{X}_n}{\sum_{k \in N} X_n^k}$ process is not a Martingale one. We will show that this technique for finding asymptotic probabilities is easier to apply than previous ones [6–8]. The property 1, as far as we know, has not been exploited in the context of finding the stationary probability measures of growing processes (nor in any other context). Conditions and more applications are discussed elsewhere.

From now on, we will consider the case when the link attachment probability law at time $n + 1$, π_{n+1}^k , depends only on the degree (and not on the in or out degrees) vector at time n (\vec{X}_n), just as it happens in the model proposed in [6].

Before computing ν_{deg}^k , let note that as in each temporal step a new node arrive, then sum of the degree vector components increase in one:

$$\sum_{k \in N} X_{n+1}^k = \sum_{k \in N} X_n^k + 1. \quad (4)$$

Replacing eq. 1 and eq. 4 in the left side of eq. 3, we get:

$$E\left(\frac{X_n^k + \delta_{D_{out}=k} + \sum_{i=1}^{D_{out}} \delta_{Y_i=k-1} - \delta_{Y_i=k}}{\sum_{k \in N} X_n^k + 1} / \frac{\vec{X}_n}{\sum_{k \in N} X_n^k} = \vec{\nu}\right) = \nu^k, \quad \forall k \in N. \quad (5)$$

From this last equation we obtain that the limit degree distribution satisfies:

$$\nu_{deg}^k = p_k + (\pi_{deg}^{k-1} - \pi_{deg}^k) E_o \quad (6)$$

where $E_o = E(D_{out}) = \sum_{k=1}^{\infty} k p_k$, and π_{deg}^j is the probability that a new link is attached to a node with degree j . Note that since we have conditioned on the fact that at time n the process is distributed according to the limit measure, the link attachment probability does not depend on time (or the state of the network). It is now necessary to specify the attachment law. Under preferential linking with attractiveness, it remains:

$$\pi_{deg}^k = \frac{(k + A)\nu_{deg}^k}{E + A} \quad (7)$$

where $E = E(D) = \sum_{k=1}^{\infty} k \nu_{deg}^k$ and A is the attractiveness [7]. Replacing eq. 7 in eq. 6, and using $E(D) = 2E_o$ (for each new node with k out-links, the total degree increases by $2k$), it is easy to conclude that the limit degree distribution (ν_{deg}^k) is given by eq. 8.

$$\nu_{deg}^k = \Psi(k + A, 3 + \delta) \sum_{j=1}^k \frac{p_j}{\Psi(j + A, 2 + \delta)} \quad (8)$$

where $\Psi(a, b) \equiv \frac{\Gamma[a]\Gamma[b]}{\Gamma[a+b]} = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ (Beta function), and $\delta = A/E_o$.

In order to compute the limit in-degree distribution, we first compute the limit joint in and out-degree distribution. If the network is distributed according to the limit measure, the probability that a randomly chosen node has j in and k out links, $\vec{D} = (D_{in}, D_{out}) = (j, k)$, is given by:

$$\nu_{in,out}^{j,k} = P(\vec{D} = (j, k)) = \lim_{n \rightarrow \infty} \frac{X_n^{j,k}}{\sum_{j,k \in N, N_o} X_n^{j,k}} \quad (9)$$

where $X_n^{h,i}$ is the number of nodes with h in-links and i out-links at time n . Since in the proposed model the link attachment depends only on the degree (eq. 7), it will be easier to study the joint degree and out-degree distribution, $\nu_{deg,out}^{j,k}$. Clearly, the joint in-out degree can be calculated from this last one, $\nu_{in,out}^{j-k,k} = \nu_{deg,out}^{j,k}$. The same procedure employed to compute ν_{deg}^k can be applied to obtain $\nu_{deg,out}^{j,k}$. In this case, the joint distribution satisfies:

$$\begin{aligned} \nu_{deg,out}^{k,k} &= p_k - \pi_{deg,out}^{k,k} E_o \\ \nu_{deg,out}^{n,k} &= \pi_{deg,out}^{n-1,k} E_o - \pi_{deg,out}^{n,k} E_o \end{aligned} \quad (10)$$

for $k \leq n \in N$. Eq. 10 is the joint version of eq. 6. These two equations contain all the information about the limit joint in-out degree distribution, being a crucial result in this paper. As before, $\pi_{deg,out}^{n,k}$ denotes the probability that a new link (from a new node) point out to an existing node with $n - k$ in-degree links and k out-degree ones. For preferential linking with attractiveness we get

$$\pi_{deg,out}^{n,k} = \frac{n + A}{2E_o + A} \nu_{deg,out}^{n,k} \quad (11)$$

In eq. 7 π_{deg}^n stands for the marginal distribution of eq. 11, $\pi_{deg}^n = \sum_{k=1}^n \pi_{deg,out}^{n,k}$. Substituting eq. 11 in eq. 10 we obtain:

$$\nu_{deg,out}^{n,k} = \frac{\Psi(n+A, 3+\delta)}{\Psi(k+A, 2+\delta)} p_k \quad (12)$$

The proportion of nodes with n links, k of which are out-links ($n-k$ in-links), depends on the out-degree distribution through two quantities: E_o and p_k .

An alternative way of obtaining the limit degree distribution (see eq. 8) is to take the marginal distribution from eq. 12, $\nu_{deg}^k = \sum_{j=1}^k \nu_{deg,out}^{k,j}$. On the other hand, using that $\nu_{in}^k = \sum_{j=1}^{\infty} \nu_{deg,out}^{k+j,j}$, is easy to compute the limit in-degree distribution:

$$\nu_{in}^k = \sum_{j=1}^{\infty} p_j \frac{\Psi(j+k+A, 3+\delta)}{\Psi(j+A, 2+\delta)}. \quad (13)$$

This is the main result of the paper, the limit in-degree distribution for arbitrary out-degree distribution. For non-random D_{out} ($p_k = \delta_{k=m}$) we recover the known result [6]. Moreover, it is possible to see that a superposition principle does not hold, either for ν_{deg}^k (eq. 8), ν_{in}^k (eq. 13), or $\nu_{in,out}^{k,j}$ (eq. 12). These cannot be written as $\nu^k = \sum_{j=1}^{\infty} p_j \mu_j^k$, where μ_j is the limit measure for $p_k = \delta_k$. The superposition principle will be valid for the three limit distributions only when the attractiveness vanishes (preferential linking). In this way, the preferential linking generalization (the inclusion of attractiveness) introduced in [7] has the advantage of enlarging the power exponent values of the degree distribution, with the drawback of losing a superposition principle. If we allow the appearance of new nodes with zero out-links ($P(D_{out} = k) = p_k$ with $k \in N_o$), then equations 12, 8, and 13 still hold after switching the initial index in the summation from 1 to 0 and taking $k \in N_o$.

For the special case of $A = 0$, it is straightforward to see that the covariance between D_{out} and D_{in} , $Cov(D_{in}, D_{out}) = E(D_{in}D_{out}) - E(D_{in})E(D_{out})$, verifies the following equation:

$$Cov(D_{in}, D_{out}) = \frac{1}{2} Cov(D, D_{out}) = Var(D_{out}). \quad (14)$$

The correlation is always positive or zero (for non random D_{out}), as it should be expected. Although it is very easy to estimate the covariance between the in and out-degree, this measure has unfortunately not been reported for any real network. It would be interesting to report this descriptive statistical measure together with the variance ($Var(D_{out}) = E(D_{out}^2) - E(D_{out})^2$) of the out-degree r.v. in order to gain some insight (see eq. 14) on whether the preferential linking law ($A = 0$) is adequate to the real growing network. This does not constitute the only non-rigorous check that can be applied to analyze the underlying link attachment law. As we will see next, we can also verify if the link attachment law is adequate by studying the relation between the tail behavior of the in-degree and the out-degree distributions. Continuing with the case $A = 0$, if the out-degree distribution has finite expectation

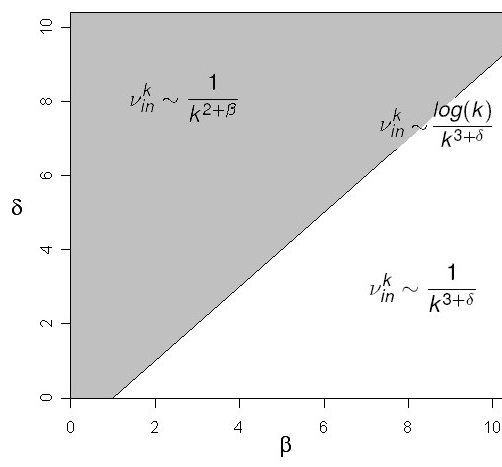


FIG. 2: Limit in-degree distribution tail under preferential linking with attractiveness for an out-degree with $p_k \sim \frac{1}{k^{2+\beta}}$ as a function of $\delta = \frac{A}{E_o}$ and β . The horizontal axis corresponds to preferential linking ($A = 0$). In the separatrix curve, $\delta = \beta - 1$, $\nu_{in}^k \sim \frac{\log(k)}{k^{3+\delta}} = \frac{\log(k)}{k^{2+\beta}}$.

and a scale invariant tail, $p_k \sim k^{-(2+\beta)}$, it is not difficult to see that the limit degree distribution and the in-degree distribution have (see eq. 8 and eq. 13) the following tail behavior:

$$\nu_{deg}^k \sim \nu_{in}^k \sim \begin{cases} k^{-(2+\beta)} & 0 < \beta < 1 \\ \log(k)k^{-3} & \beta = 1 \\ k^{-3} & \beta > 1 \end{cases} \quad (15)$$

Eq. 15 is our third main result: for a scale invariant tail out-degree distribution with finite variance, $p_k \sim k^{-(2+\beta)}$, the limit in-degree distribution also has a scale invariant tail, $\nu_{in} \sim k^{-\alpha}$. Moreover, for $0 < \beta < 1$, α is equal to the out-degree exponent, which explains why in so many real networks the in and out exponents are very similar, taking values in a range from 2 to 3. In the case $\beta > 1$, $\alpha = 3$, regardless of the value of β . For the frontier case (finite/infinite variance) of $\beta = 1$, the limit distribution decays at a slower rate than k^{-3} . Precisely, it decays as $\nu_{in} \sim \log(k)k^{-3}$. In the general case of preferential linking with attractiveness for $p_k \sim k^{-(2+\beta)}$, the regimes are similar to the non-attractiveness case. The only difference being that there is now a separatrix curve between them, as it is shown in Fig. 2. The behavior is regulated by $\delta \equiv A/E_o$ and β . For $\delta > 1 + \beta$ the limit out degree $\nu_{in}^k \sim k^{-(2+\beta)}$, and in this case the (in) degree distribution has exactly the same tail as the out-degree, even for large β . For $\delta < 1 + \beta$, ν_{in}^k behaves as $k^{-(3+\delta)}$. Finally on the separatrix curve, $\delta = 1 + \beta$, the behavior is given by $\log(k)k^{-(3+\delta)}$.

For out-degree distributions with exponential tails, as a geometric, Poisson, or finite range distributions, it is easy to see that the in-degree distribution $\nu_{in}^k \sim k^{-(3+\delta)}$. This is also a very interesting result: in [9] they show that in the context of the PRL citation network the out-degree distribution has an exponential decay, and the in-degree one has a power law tail with α near 3, just as described before. We remark that an empirical growing network with

exponential out-degree distribution tail and scale invariant in-degree one with power exponent less than 3, cannot have a preferential linking with attractiveness mechanism. It is thus clear that even when preferential linking is an accepted mechanism of link attachment, it is necessary to study [8] alternative types. For example, for the uniform attachment law:

$$\pi_{deg,out}^{n,k} = \nu_{deg,out}^{n,k} \quad (16)$$

by means of the same technology (replacing $\pi_{deg,out}^{n,k}$ in eq. 10) we obtain:

$$\begin{aligned} \nu_{deg}^k &= \frac{1}{1+E_o} \sum_{j=0}^k p_j \left(\frac{E_o}{1+E_o} \right)^{k-j} \\ \nu_{in}^k &= \frac{1}{1+E_o} \left(\frac{E_o}{1+E_o} \right)^k \end{aligned} \quad (17)$$

Note that, the limit in degree distribution depends only on E_o (and not on p_k), and decays exponentially fast. For an out-degree with $p_k \sim k^{-(2+\beta)}$, ν_{deg}^k behaves as $k^{-(2+\beta)} f(k)^{-1}$, where $f(k)$ is an increasing function of k that grows much slower than $\log(k)$. It is important to remark that for empirical (finite) networks, the $f(k)^{-1}$ term will be very difficult to discriminate ($f(k)$ grows at a rate slower than $\log(\log(k))$). This behavior may be hard to “separate” from $\nu_{deg}^k \sim k^{-(2+\beta)}$, but the in-degree distribution will sort out any possible confusion about the attachment linking law.

In summary, under the model presented here, we discussed: 1) a way of computing the stationary probability of the process, 2) the limit joint in and out degree distributions for arbitrary out degree distribution and link attachment probability law. From the joint distribution we obtain some of its derivatives, who explain the in/out degree relation reported in a variety of real networks.

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