DEPARTAMENTO DE MATEMÁTICA
DOCUMENTO DE TRABAJO

"A coagulation estimate for inertial particles"

Inés Armendáriz y James R. Norris

D.T.: N° 49 Junio 2007

Vito Dumas 284, (B1644BID) Victoria, Pcia. de Buenos Aires, Argentina
Tel: 4725-7053 Fax: 4725-7010
Email: matematica@udesa.edu.ar

A coagulation estimate for inertial particles

Inés Armendáriz and James R. Norris

Abstract

This paper establishes a key convergence result for coagulation probabilities of particles which behave as Brownian motions on the scale of their mean square displacement but as integrated Ornstein–Uhlenbeck processes on the scale of the particle radius. The result is a first step towards the rigorous derivation of an analogue of Smoluchowski's coagulation equation for inertial particles.

1 Introduction

The phenomenon of pairwise coagulation in large particle systems appears frequently in scientific models describing the evolution of clusters of basic particles. The rate at which clusters coagulate depends on the characteristics of the model. When the effect of spatial fluctuations in the mass density is negligible, this rate will be a function of the mass and other details of each cluster, and it will be spatially homogeneous. On the other hand, in many physical models the determining factor for coagulation is proximity, and the coagulation rate will thus depend on the position of the incoming clusters.

Over the past few years, several spatial models that include stochastic dynamics of coagulation have been studied (see [3], [5], [6], [1]). Here, clusters move freely in space until any two of them meet at less then a prescribed distance, which typically goes to zero as the particle number increases. At this time, the clusters may coagulate at some positive rate, in which case they will be replaced by a single cluster, and the evolution of the system is resumed.

The hardcore case, where particles coagulate on collision, has been much less studied. These models arise quite naturally: for instance in the case of microscopic particles undergoing molecular bombardment, we might describe the free evolution as Brownian motion and specify that a pair of particles coagulates the first time it meets, which leads to a spatial generalization of the original Smoluchowski's coagulation equation [11]. The technical difficulty with this model lies in that the coagulation events are completely deterministic once the cluster paths are known. The problem was first studied by Lang and Xuan-Xanh [7] in the case of discrete mass, constant diffusivity and when the radius of each particles does not depend on the mass it carries. In a forthcoming paper [9], Norris removes these restrictions, treating the case of variable coefficients and both discrete and continuous mass.

We are now interested in replacing the idealization that the free motion of each particle is Brownian, by the more physically reasonable assumption that it performs an integrated Ornstein–Uhlenbeck process at the microscopic scale given by its radius. That is, we allow the particles to have velocities.

The starting point of the analysis is to estimate the first passage time of such an inertial particle into a small radius ball. In terms of the particle system, this corresponds to establishing the probability that a particular pair of particles collides in finite time. The ball radius will then be a function of the particle number, converging to zero at a faster rate than the velocities of the particles diverge to infinity.

The problem of computing the probability that the first passage time into a ball of small radius occurs before a given time was first addressed by Spitzer [12] in 1958, in the case when the particle performs a planar Brownian motion. Several years later, in 1986, Le Gall [8] deviced a method to compute this probability for a large family of diffusions, in dimensions $d \geq 2$. The fact that in our case the dynamics are not Markovian prevents us from applying these techniques, and we are forced to develop an ad–hoc method.

We introduce the model and describe the main results of the paper in Section 2. The process behaves as Brownian motion on the scale of the mean square displacement; this is used in Section 3 to estimate the first passage time into a ball which is large relative to the particle radius. We study an auxiliary linear process in Section 4, and finally obtain the first passage time estimate for the inertial particle in Section 5.

2 Notation and results

Let $d \geq 3$. Given $\sigma > 0$ and a standard, d–dimensional Brownian motion $W(\cdot)$, $W(0) = 0$, consider the stationary Ornstein–Uhlenbeck process determined by

$$dZ = \sigma dW - Z\,ds, \quad Z(0) \sim N\Big(0, \frac{\sigma^2}{2}\,I_d\Big),$$

I_d the d–dimensional identity matrix, and define the integrated process

$$X(t) = x_0 + \sqrt{\beta}\int_0^t Z(\beta u)\,du,$$

where $x_0 \in \mathbb{R}^d, x_0 \neq 0$ and β is a positive parameter. Then $X(\cdot)$ describes the motion of a particle of unitary mass in a force field $\sigma\beta\dot{W}$, where $\dot{W}$ is the standard Gaussian white noise in $\mathbb{R}^d$. The particle experiences friction proportional to the velocity. The position $X(\cdot)$ hence follows the Newton law

$$\ddot{X} = \sigma\beta\dot{W} - \beta\dot{X}, \qquad X(0) = x_0 \in \mathbb{R}^d, \qquad \dot{X}(0) \sim N\Big(0, \frac{\sigma^2}{2}\,I_d\Big).$$

We start by deriving an alternative expression for X.

Lemma 1. *Let $U(\cdot)$ be the standard Brownian motion given by*

$$U(s) = \frac{W(\beta s)}{\sqrt{\beta}}, \;\; s \geq 0.$$

Then

$$Z(\beta t) = e^{-\beta t} Z(0) + \sigma \sqrt{\beta}\, e^{-\beta t} \int_0^t e^{\beta u} dU(u) \tag{2.1}$$

and

$$X(t) = x_0 + \frac{(1 - e^{-\beta t})}{\sqrt{\beta}}\, Z(0) + \sigma\, U(t) - \sigma\, e^{-\beta t} \int_0^t e^{\beta s}\, dU(s). \tag{2.2}$$

Proof. Integrate the SDE satisfied by Z to obtain the representation

$$Z(t) = e^{-t} Z(0) + \sigma\, e^{-t} \int_0^t e^{s} dW(s),$$

change time by $t \to \beta t$ and make the substitution $s = \beta u$ is the integral term to get (2.1). Replacing this expression in the definition of X, we have

$$\begin{aligned}
X(t) &= x_0 + \sqrt{\beta} \int_0^t Z(\beta s)\, ds \\
&= x_0 + \sqrt{\beta} \int_0^t \left(e^{-\beta s} Z(0) + \sigma \sqrt{\beta}\, e^{-\beta s} \int_0^s e^{\beta l} dU(l) \right) ds \\
&= x_0 + \frac{1}{\sqrt{\beta}}\, (1 - e^{-\beta t})\, Z(0) + \sigma \int_0^t \int_0^s \beta e^{\beta l}\, dU(l)\, e^{-\beta s}\, ds \\
&= x_0 + \frac{1}{\sqrt{\beta}}\, (1 - e^{-\beta t})\, Z(0) + \sigma \int_0^t (e^{-\beta l} - e^{-\beta t})\, e^{\beta l}\, dU(l) \\
&= x_0 + \frac{1}{\sqrt{\beta}}\, (1 - e^{-\beta t})\, Z(0) + \sigma\, U(t) - \sigma\, e^{-\beta t} \int_0^t e^{\beta l}\, dU(l),
\end{aligned}$$

as claimed. □

Let now $\epsilon > 0$. Our first goal is to estimate the probability that X visits a ball of radius ϵ centered at the origin before a fixed final time T, as the parameters ϵ and β tend to 0 and ∞ in that order.

As discussed in the introduction, the techniques developed by Le Gall in [8] cannot be applied in our case since $X(\cdot)$ is not Markov. Instead, the expression in (2.2) suggests the strategy to follow. The first step will be to compute the probability that the process hits a relatively big ball of radius $g(\beta)$ prior to T, where g is chosen so that the only relevant terms in (2.2) are $x_0 + \sigma\, U(t)$. The problem is thus reduced to the case of standard Brownian motion, where the answer is well known. Then, conditioned on this event, we will estimate the probability that it enters the ball of radius ϵ by considering a piecewise linear approximation of X.

The first part of the plan is carried out in the following proposition. As will become clear in the proof, the condition on $g(\beta)$ is that $h(\beta)$ given by $h(\beta) = \sqrt{\beta} g(\beta)$ satisfies

$$\beta \exp\big[- h^2(\beta) \big]\, \frac{1}{g(\beta)^{d-2}} \longrightarrow 0$$

in the limit as $\beta \to \infty$. We choose to work with $g(\beta) = \log\beta/\sqrt{\beta}$.

Proposition 1.

$$P\left[\inf_{0\le t\le T} \|X(t)\| \le \frac{\log\beta}{\sqrt{\beta}}\right] = \left(\frac{d}{2}-1\right)\,\omega_d\,\sigma^2\,\left(\frac{\log\beta}{\sqrt{\beta}}\right)^{d-2}\int_0^T p_s^\sigma(x_0,0)\,ds + o\left[\left(\frac{\log\beta}{\sqrt{\beta}}\right)^{d-2}\right],$$

where $p_s^\sigma(x,y)$ denotes the Gaussian transition density

$$p_s^\sigma(x,y) = (2\pi\sigma^2 s)^{-d/2}\,\exp\left[-\frac{1}{2\sigma^2 s}\|x-y\|^2\right]$$

and ω_d is the volume of the d–dimensional unit sphere.

Notice that the condition on $g(\beta)$ is in particular satisfied by $g(\beta)=\delta$, for any positive constant δ. The proof of the version of Proposition 1 corresponding to this choice of g relies on showing that

$$P\left[\sup_{0\le t\le T} \|e^{-\beta t}\int_0^t e^{\beta s}\,dU_s\| \ge \delta\right] = o\left[\delta^{d-2}\right].$$

Combined with (2.2), this implies that the process $X(\cdot)$ converges in distribution to a Brownian motion. In fact, a much stronger convergence result holds. In [4] (see also the references therein), Freidlin proves that

$$\lim_{\beta\to\infty}\max_{0\le t\le T}\|X(t)-\sigma W(t)\| = 0 \quad a.s..$$

Suppose now that the process has reached the ball of radius $\log\beta/\sqrt{\beta}$ at time $t_0 < T$ for the first time, $X(t_0) = (\log\beta/\sqrt{\beta})\,y_0$, $\|y_0\|=1$. We then need to compute the probability that

$$\|X(t_0+s)\| = \left\|\frac{\log\beta}{\sqrt{\beta}}\,y_0 + \frac{1}{\sqrt{\beta}}\int_0^{\beta s} Z(\beta t_0+u)\,du\right\| \le \epsilon$$

for some time $0\le s\le T-t_0$.

Let us first simplify the notation by rescaling space and time. Define

$$Y(t) = y_0 + \int_0^t Z(\beta t_0 + u\log\beta)\,du, \quad 0\le t\le \frac{\beta}{\log\beta}(T-t_0).$$

A change of variables in the integral term of X yields

$$X(t_0+s) = \frac{\log\beta}{\sqrt{\beta}}\,Y\left(\frac{\beta}{\log\beta}\,s\right), \quad 0\le s\le T-t_0\,,$$

and the task is now to estimate the probability that $Y(\cdot)$ visits the ball of radius $\tilde{\epsilon}$ before time $(\beta/\log\beta)(T-t_0)$, where

$$\tilde{\epsilon} = \frac{\sqrt{\beta}}{\log\beta}\,\epsilon.$$

This plan will be carried out for a piecewise linear approximation $\tilde{Y}$ of Y in Lemmas 2, 3, and 4, Section 4. Combining the latter with a result measuring the level of accuracy of the approximation $Y \sim \tilde{Y}$ (Lemma 5, Section 5), we obtain a first order estimate.

Proposition 2.

$$P\left[\inf_{0\le t\le T}\|X(t)\| \le \epsilon\right] = C_d\,\sqrt{\beta}\,\epsilon^{d-1}\,\sigma\int_0^T p_s^\sigma(0,x_0)\,ds + o\left[\sqrt{\beta}\epsilon^{d-1}\right]$$

with

$$C_d = \frac{1}{\sqrt{2}}\,\big(\frac{d}{2}-1\big)\,\omega_{d-1}\,\omega_d\left(\int_0^\infty p_s^1(0,y)\,ds\right)\left(\int_{\mathbb{R}^d}\|z\|\,p_1^1(0,z)\,dz\right),$$

$\|y\| = 1$ *and* ω_d *the volume of the* d*–dimensional unit sphere.*

Throughout the article, C will denote a positive constant. Unless we are particularly interested in keeping track of its growth or dependence on the parameters, we will use the same letter C to denote constants on consecutive lines which may be different, or constants appearing in totally unrelated computations.

3 Brownian first passage time

This section contains the

Proof of Proposition 1. Let $t \ge 0$. Due to (2.2), $X(t)$ may be decomposed as

$$X(t) = A(t) + B(t),$$

where $A(\cdot)$ and $B(\cdot)$ are defined by

$$A(t) = x_0 + \sigma\,U(t)$$

and

$$B(t) = \frac{(1-e^{-\beta t})}{\sqrt{\beta}}\,Z(0) - \sigma\,e^{-\beta t}\int_0^t e^{\beta s}\,dU(s).$$

Let $r = r(\beta) = (\log\beta)^{-1/8}$. We have the following inclusions

$$\left\{ \inf_{0\le t\le T} \|A(t)\| \le (1-r)\frac{\log\beta}{\sqrt{\beta}} \quad \text{and} \quad \sup_{0\le t\le T} \|B(t)\| \le r\,\frac{\log\beta}{\sqrt{\beta}} \right\}$$
$$\subseteq \left\{ \inf_{0\le t\le T} \|X(t)\| \le \frac{\log\beta}{\sqrt{\beta}} \right\} \tag{3.1}$$
$$\subseteq \left\{ \inf_{0\le t\le T} \|A(t)\| \le (1+r)\frac{\log\beta}{\sqrt{\beta}} \quad \text{or} \quad \sup_{0\le t\le T} \|B(t)\| \ge r\,\frac{\log\beta}{\sqrt{\beta}} \right\}.$$

We will show that there exists a positive constant Γ such that

$$P\left[\sup_{0\le t\le T} \|B(t)\| \ge r\,\frac{\log\beta}{\sqrt{\beta}} \right] \le \Gamma\beta\,T\exp\left\{-(\log\beta)^{3/2}\right\} \tag{3.2}$$
$$\le o\left[\left(\frac{\log\beta}{\sqrt{\beta}}\right)^{d-2} \right].$$

Suppose that (3.2) holds. Then by (3.1)

$$P\left[\inf_{0\le t\le T} \|A(t)\| \le (1-r)\frac{\log\beta}{\sqrt{\beta}} \right] - \Gamma\,\beta\,T\exp\left\{-(\log\beta)^{3/2}\right\}$$
$$\le P\left[\inf_{0\le t\le T} \|X(t)\| \le \frac{\log\beta}{\sqrt{\beta}} \right] \tag{3.3}$$
$$\le P\left[\inf_{0\le t\le T} \|A(t)\| \le (1+r)\frac{\log\beta}{\sqrt{\beta}} \right] + \Gamma\,\beta\,T\exp\left\{-(\log\beta)^{3/2}\right\}.$$

On the other hand, Le Gall showed in [8] that

$$P\left[\inf_{0\le t\le T} \|A(t)\| \le (1\pm r)\,\frac{\log\beta}{\sqrt{\beta}} \right] =$$
$$= \left(\frac{d}{2}-1\right)\omega_d\,\sigma^2\,(1\pm r)^{d-2}\left(\frac{\log\beta}{\sqrt{\beta}}\right)^{d-2}\int_0^T p_s(x_0,0)\,ds$$
$$+\,o\left[\left(\frac{(1\pm r)\log\beta}{\sqrt{\beta}}\right)^{d-2}\right].$$

The proposition will then follow from (3.3) and the choice of r.

It remains to prove (3.2). Equivalently, we need to show that

$$P\left[\sup_{0\le t\le T} \left| e^{-\beta t}\int_0^t e^{\beta s}\,dW_1(s)\right| \ge \frac{r}{d}\,\frac{\log\beta}{\sqrt{\beta}} \right] < \frac{1}{d}\,\Gamma\,\beta\,T\exp\left\{-(\log\beta)^{3/2}\right\},$$

where W_1 is a standard 1–dimensional Brownian motion. We will achieve this by means of the formulas for the exponential martingales of a diffusion.

We split $[0,T]$ into $\lceil\beta T\rceil$ subintervals having length less than or equal to $1/\beta$, where given $l\in\mathbb{R}$, $\lceil l\rceil$ denotes the smallest integer larger than or equal to

l. Let t_j, $0 \le j < \lceil \beta T \rceil$ be the left endpoints of these subintervals. Fix j, and denote by

$$\delta = \frac{r}{d}\frac{\log \beta}{\sqrt{\beta}}\,, \quad \lambda = \lambda_j = \frac{1}{\delta}\,[\log \beta]^{3/2}\, \exp[-\beta t_j].$$

We obtain

$$\begin{aligned}
P\left[\sup_{t_j \le t \le t_{j+1}} \left| e^{-\beta t}\int_0^t e^{\beta s}\, dW_1(s)\right| \ge \delta\right] &\le P\left[\sup_{t_j \le t \le t_{j+1}} \left| \int_0^t e^{\beta s}\, dW_1(s)\right| \ge \delta e^{\beta t_j}\right] \\
&\le 2P\left[\sup_{t_j \le t \le t_{j+1}} \exp\left\{\lambda \int_0^t e^{\beta s}\, dW_1(s) - \frac{\lambda^2}{2}\int_0^t e^{2\beta s}\, ds\right\}\right. \\
&\qquad\qquad \left. \ge \exp\left\{\delta\lambda\, \exp[\beta t_j] - \frac{\lambda^2}{4\beta}\exp[2\beta t_j + 2]\right\}\right] \\
&\le C\, \exp\left\{-(\log \beta)^{3/2}\right\}, \qquad (3.4)
\end{aligned}$$

after applying Doob's inequality to the exponential martingale

$$\exp\left\{\lambda \int_0^t e^{\beta s}\, dW_1(s) - \frac{\lambda^2}{2}\int_0^t e^{2\beta s}\, ds\right\}$$

and replacing δ and λ by their values. The positive constant C in the last line can be chosen uniformly in β, for β large enough. We finally add over $0 \le j < \lceil \beta t \rceil$ in (3.4) to conclude (3.2). $\square$

4 An associated linear process

In order to prove Proposition 2, we will replace the process Y by a piecewise linear approximation $\tilde{Y}$ for which we are able to compute the probability that the first passage time into the ball of radius $\tilde{\epsilon}$ occurs prior to $(\beta/\log\beta)(T-t_0)$, where we recall that t_0 denotes the first time the process $X(\cdot)$ reaches the ball of radius $\log\beta/\sqrt{\beta}$. It is therefore implicitly assumed that this event has occurred: in the next four lemmas all probabilities are initially computed given

$$t_0 \le T, \qquad Z(\beta t_0) \qquad \text{and} \qquad y_0 = \frac{\sqrt{\beta}}{\log \beta} X(t_0),$$

until the conditioning is eventually removed.

Consider a regular partition of

$$\left[0, \frac{\beta}{\log \beta}(T - t_0)\right]$$

into intervals $[s_i, s_{i+1}]$ of length

$$h = h(\tilde{\epsilon}) = \tilde{\epsilon}^{\frac{2}{3}+\mu}, \qquad \mu > 0\,,$$

and define

$$\tilde{Y}(u) = Y(s_i) + (u - s_i) Z\left(\beta t_0 + s_i \log \beta\right)$$

if

$$s_i \leq u < s_{i+1}, \quad 0 \leq i < \left\lceil \frac{\beta}{\log \beta} \frac{(T - t_0)}{h} \right\rceil .$$

We are interested in setting the steps of the piecewise linear process as large as possible while making sure that the processes $Y(\cdot)$ and $\tilde{Y}(\cdot)$ stay within a sufficiently small distance. The choice of h is hence motivated by the proof of Lemma 5 in Section 5, where we show that the error resulting from the substitution $\tilde{Y}(\cdot)$ for $Y(\cdot)$ is neglibible.

Define now

$$Y_i = Y(s_i) = \tilde{Y}(s_i), \quad Z_i = Z(\beta t_0 + s_i \log \beta), \quad 0 \leq i < \left\lceil \frac{1}{h} \frac{\beta}{\log \beta} (T - t_0) \right\rceil .$$

As discussed before, we assume at this point that the initial values $\big(Z(\beta t_0), y_0\big)$ of the $2d$–dimensional random vector

$$\Big(Z(\beta t_0 + \cdot \, \log \beta), Y(\cdot) \Big)$$

are given. Standard computations for the Ornstein Uhlenbeck process show that $Z(\beta t_0 + t \log \beta)$ is Gaussian with mean $e^{-t \log \beta} \, Z(\beta t_0)$ and covariance matrix

$$\Gamma(s,t) = \frac{\sigma^2}{2} \left[e^{-|t-s| \log \beta} - e^{-(t+s) \log \beta} \right] I_d ,$$

I_d the d–dimensional identity matrix.

By definition,

$$Y(t) = y_0 + \int_0^t Z(\beta t_0 + u \log \beta) \, du ,$$

from where it follows that (Z_i, Y_i) is Gaussian with mean

$$m_i = (z_i, y_i) = \left(e^{-s_i \log \beta} \, Z(\beta t_0), \, y_0 + \frac{(1 - e^{-s_i \log \beta})}{\log \beta} \, Z(\beta t_0) \right)$$

and covariance matrix given by the block matrix

$$\Gamma_i = \begin{pmatrix} a_i \, I_d & b_i \, I_d \\ b_i \, I_d & c_i \, I_d \end{pmatrix}$$

with determinant $D_i = d_i^d = (a_i c_i - b_i^2)^d$, where

$$\begin{aligned} a_i &= \frac{\sigma^2}{2} \left[1 - e^{-2 s_i \log \beta} \right] \\ b_i &= \frac{\sigma^2}{2} \frac{1}{\log \beta} \left[1 - 2 e^{-s_i \log \beta} + e^{-2 s_i \log \beta} \right] = \frac{\sigma^2}{2} \frac{1}{\log \beta} \left[1 - e^{-s_i \log \beta} \right]^2 \end{aligned}$$

and

$$c_i = \sigma^2 \left[\frac{s_i}{\log\beta} - \frac{2}{(\log\beta)^2}\left(1 - e^{-s_i \log\beta}\right) + \frac{1}{2(\log\beta)^2}\left(1 - e^{-2s_i\log\beta}\right)\right].$$

The probability density function of (Z_i, Y_i) then becomes

$$p_i(z_i, y_i; z, y) = \frac{1}{(2\pi)^d\sqrt{D_i}}\, \exp\left[-\frac{1}{2}\,(z - z_i, y - y_i)^t\, \Gamma_i^{-1}(z - z_i, y - y_i)\right],$$

which can be rearranged into

$$\begin{aligned} p_i(z_i, z)\, p_i(y_i, y | z_i, z) = & \frac{1}{(2\pi\, a_i)^{d/2}}\, \exp\left[-\frac{1}{2\, a_i}\|z - z_i\|^2\right] \\ & \times\, \frac{1}{(2\pi(d_i/a_i))^{d/2}}\, \exp\left[-\frac{1}{2}\frac{a_i}{d_i}\left\|y - y_i - \frac{b_i}{a_i}(z - z_i)\right\|^2\right]. \end{aligned} \tag{4.1}$$

The relevance of this representation lies in that the second line is the density $p_i(y_i, y|z_i, z)$ of Y_i conditioned on the value achieved by Z_i, whereas the expression on the first line is the density of Z_i, $p_i(z_i, z)$.

4.1 An upper bound

Our purpose here is to establish, to first order, an upper bound to the conditional probability that the process $\tilde{Y}(\cdot)$ enters the ball of radius $\tilde{\epsilon}$ before time $(\beta/\log\beta)(T - t_0)$.

We clearly have

$$\begin{aligned} P & \left[\inf_{0 \le t \le \frac{\beta}{\log\beta}(T - t_0)} \|\tilde{Y}(t)\| \le \tilde{\epsilon}\,\Big|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right] \\ & \le \sum_{0 \le i < \lceil \frac{1}{h}\frac{\beta}{\log\beta}(T - t_0)\rceil} P\left[\inf_{s_i \le t < s_{i+1}} \|\tilde{Y}(t)\| \le \tilde{\epsilon}\,\Big|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right], \end{aligned} \tag{4.2}$$

and it will suffice to compute the right side of (4.2).

Denote by $\mathcal{A}_i$ the event that $\tilde{Y}$ visits the $\tilde{\epsilon}$–ball centered at the origin over the time interval $[s_i, s_{i+1})$,

$$\mathcal{A}_i = \left\{\inf_{s_i \le t < s_{i+1}} \|\tilde{Y}(t)\| \le \tilde{\epsilon}\right\}$$

Let $\mathcal{C}_i$ be the right, convex cylinder with axis $-hZ_i$, radius $\tilde{\epsilon}$ and semi–spherical caps of radius $\tilde{\epsilon}$, open at the end associated to $-hZ_i$. Then $\mathcal{A}_i$ is the event that

Y_i belongs to $\mathcal{C}_i$,

$$\mathcal{A}_i = \mathcal{A}_i^1 \cup \mathcal{A}_i^2 = \Big\{ 0 \leq -\langle Y_i, Z_i \rangle < h \|Z_i\|^2,\ \ \|\pi_{Z_i^\perp}(Y_i)\| \leq \tilde{\epsilon} \Big\} \\ \cup \Big\{ \|Y_i\| \leq \tilde{\epsilon} \ \text{ or } \ \|Y_i + hZ_i\| < \tilde{\epsilon} \Big\}$$

if $\pi_{Z_i^\perp}$ denotes the projection onto the subspace orthogonal to Z_i.

We will first control the probability of the sets of type $\mathcal{A}_i^1$. Combining (4.1) with the definition of $\mathcal{A}_i^1$, we have

$$P\big[\mathcal{A}_i^1 \,|\, \{t_0,\, Z(\beta t_0),\, y_0\}\big] = \\ = \int_{\mathbb{R}^d} p_i(z_i, z) \times \left[\int_{\mathbb{R}^d} 1_{\{0 \leq -\langle z, y\rangle \leq h\|z\|^2\}}\, 1_{\left\{\|\pi_{z^\perp}(y)\| \leq \tilde{\epsilon}\right\}}\, p_i(y_i, y | z_i, z)\, dy \right] dz$$

Let $\mathcal{B}(z^\perp)$ be an orthonormal basis of the $(d-1)$–dimensional subspace $z^\perp$. We write y, y_i and z_i as a linear combination of the vectors in the basis $\{z/\|z\|,\, \mathcal{B}(z^\perp)\}$ and change coordinates in the y–integral, to obtain

$$\begin{aligned} P\big[\mathcal{A}_i^1 \,|\, \{t_0,\, Z(\beta t_0),\, y_0\}\big] &= \int_{\mathbb{R}^d} \frac{1}{(2\pi a_i)^{d/2}} \exp\left[-\frac{1}{2a_i} \|z - z_i\|^2 \right] \\ \times \Bigg(\int_{-h\|z\|}^{0} &\frac{1}{\sqrt{2\pi(d_i/a_i)}} \\ &\times \exp\left[-\frac{a_i}{2d_i} \left| y^1 - \Big\langle y_i, \frac{z}{\|z\|} \Big\rangle - \frac{b_i}{a_i} \Big(\|z\| - \Big\langle z_i, \frac{z}{\|z\|} \Big\rangle \Big) \right|^2 \right] dy^1 \\ &\times \int_{\substack{y^{d-1} \in \mathbb{R}^{d-1} \\ \|y^{d-1}\| \leq \tilde{\epsilon}}} \frac{1}{\big(2\pi(d_i/a_i)\big)^{\frac{d-1}{2}}} \\ &\qquad \times \exp\left[-\frac{a_i}{2d_i} \left\| y^{d-1} - \pi_{z^\perp}(y_i) + \frac{b_i}{a_i} \pi_{z^\perp}(z_i) \right\|^2 \right] dy^{d-1} \Bigg)\, dz. \end{aligned} \tag{4.3}$$

In the last line we have kept the notation $\pi_{z^\perp}(y_i)$ and $\pi_{z^\perp}(z_i)$ to denote the $(d-1)$–last coordinates of these vectors in the basis $\{\frac{z}{\|z\|},\, \mathcal{B}(z^\perp)\}$.

Lemma 2.

$$\sum_{0 \leq i < \lceil \frac{1}{h} \frac{\beta}{\log \beta}(T - t_0) \rceil} P\left[\mathcal{A}_i^1 \,\Big|\, \{t_0,\, Z(\beta t_0),\, y_0\} \right] \\ \leq c_d\, \frac{1}{\sigma}\, \tilde{\epsilon}^{\,d-1} \log \beta \ + G\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big).$$

Here

$$c_d = \frac{1}{\sqrt{2}}\, \omega_{d-1} \left(\int_0^\infty p_s^1(0, y)\, ds \right) \left(\int_{\mathbb{R}^d} \|z\|\, p_1^1(0, z)\, dz \right),$$

$\|y\| = 1$, *and* G *is an integrable random variable such that*

$$E\Big[\big|\, G\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big)\big|\Big] = o\big[\sqrt{\beta}\epsilon^{d-1}\big].$$

Proof. Let us first consider $s_i \leq 1/\log\beta$. Note that the coefficients a_i, b_i, and d_i vanish with orders 1, 2 and 4 at the origin, respectively. It will thus be necessary to control the contribution from the values of (Z_i, Y_i) such that the exponents in the second and third lines of (4.3) are small. Denote by

$$\alpha_i = d_i/a_i \qquad \text{and} \qquad \gamma_i = b_i/a_i.$$

We will split $\mathcal{A}_i^1$ according to whether

$$\|Y_i - y_i - \gamma_i(Z_i - z_i)\| \leq \eta_i \qquad \text{or} \qquad \|Y_i - y_i - \gamma_i(Z_i - z_i)\| > \eta_i,$$

where η_i is given by

$$\eta_i = (\log\beta)^{1/4} s_i\,.$$

On the first of these sets we get

$$\begin{aligned}
&\Big\{\|Y_i - y_i - \gamma_i(Z_i - z_i)\| \leq \eta_i\Big\} \cap \mathcal{A}_i^1 \\
&\qquad \subseteq \Big\{1 - \Big[\big(\eta_i + \tilde{\epsilon}\big) + \big(s_i + h\big)\|Z(\beta t_0)\|\Big] \leq \big(h + \gamma_i\big)\|Z_i - z_i\|\Big\} \cap \mathcal{A}_i^1 \\
&\qquad \subseteq \Big\{\frac{1 - 2\eta_i\big(1 + \|Z(\beta t_0)\|\big)}{2s_i} \leq \|Z_i - z_i\|\Big\} \cap \mathcal{A}_i^1\,.
\end{aligned}$$

Next, we consider the cases when

$$\|Z(\beta t_0)\| \leq \frac{1}{16\,\eta_i} \qquad \text{and} \qquad \|Z(\beta t_0)\| > \frac{1}{16\,\eta_i}$$

separately, and apply the analogue of formula (4.3) on the set

$$\mathcal{A}_i^1 \cap \Big\{\|Y_i - y_i - \gamma_i(Z_i - z_i)\| \leq \eta_i\Big\}$$

to obtain

$$\begin{aligned}
P\left[\mathcal{A}_i^1 \cap \|Y_i - y_i - \gamma_i(Z_i - z_i)\| \leq \eta_i \,\middle|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right]& \\
\leq C_d\, h\, \tilde{\epsilon}^{\,d-1}\, \frac{1}{\alpha_i^{d/2}}\, 1_{\left\{\|Z(\beta t_0)\| \leq \frac{1}{16\,\eta_i}\right\}} \int_{\frac{1}{4s_i} \leq \|z - z_i\|} \|z\|\, p_i(z_i, z)\, dz& \\
+ C_d\, h\, \tilde{\epsilon}^{\,d-1}\, \frac{1}{\alpha_i^{d/2}}\, 1_{\left\{\|Z(\beta t_0)\| > \frac{1}{16\,\eta_i}\right\}} \int_{\mathbb{R}^d} \|z\|\, p_i(z_i, z)\, dz.&
\end{aligned}$$

Since $a_i \sim s_i \log\beta \leq 1$, the second line above is bounded by

$$C_d\, h\, \tilde{\epsilon}^{\,d-1}\, \frac{1}{\alpha_i^{d/2}}\, 1_{\left\{\|Z(\beta t_0\| \leq \frac{1}{16\,\eta_i}\right\}} \left(1 + \|z_i\|\right) \exp\left[-\frac{C}{s_i^3\, \log\beta}\right]. \tag{4.4}$$

We replace $z_i = \exp[-s_i \log\beta]\, Z(\beta t_0)$ and use (4.4) to compute

$$\begin{aligned} &P\left[\mathcal{A}_i^1 \cap \|Y_i - y_i - \gamma_i(Z_i - z_i)\| \le \eta_i \,\middle|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right] \\ &\quad \le C_d\, h\, \tilde\epsilon^{\,d-1}\, \frac{1}{\alpha_i^{d/2}} \left[\left(1 + \frac{1}{s_i}\right) \exp\left(-\frac{C}{s_i^3 \log\beta}\right) \right. \\ &\qquad\qquad \left. + 1_{\left\{\|Z(\beta t_0)\| > \frac{1}{16\,\eta_i}\right\}} \left(\sqrt{s_i \log\beta} + \|Z(\beta t_0)\|\right) \right] \\ &\quad \le C_d\, h\, \tilde\epsilon^{\,d-1} \left[\exp\left(-\frac{C}{s_i^2}\right) + g\big(s_i, \|Z(\beta t_0)\|\big) \right]. \end{aligned} \tag{4.5}$$

The positive random variable

$$g\big(s_i, \|Z(\beta t_0)\|\big) = \frac{1}{\alpha_i^{d/2}}\, 1_{\left\{\|Z(\beta t_0)\| > \frac{1}{16\,\eta_i}\right\}} \left(\sqrt{s_i \log\beta} + \|Z(\beta t_0)\|\right)$$

satisfies

$$\begin{aligned} &E\left[g\big(s_i, \|Z(\beta t_0)\|\big)\, 1_{\{t_0 \le T\}}\right] \\ &\quad \le \frac{2}{\alpha_i^{d/2}} \left(1 + E\left[\sup_{0\le t\le T} \|Z(\beta t)\|^2\right]\right)^{1/2} P\left[\sup_{0\le t\le T} \|Z(\beta t)\| \ge \frac{1}{16\,\eta_i}\right]^{1/2} \\ &\quad \le C(\sigma, T)\, \beta^2 \exp\left[-\frac{C}{\eta_i^{3/2}}\right]. \end{aligned} \tag{4.6}$$

In order to derive this inequality we estimated the expectation on the second line by integrating by parts in (2.1) and then taking the supremum term by term. The bound on $P\big[\sup \|Z(\beta t)\| \ge 1/(16\,\eta_i)\big]^{1/2}$ follows by the same argument applied in (3.4) to obtain (3.2), if instead of taking δ and λ as defined there, we set

$$\delta^i = \frac{1}{16\sqrt{\beta}\,\eta_i} \qquad \text{and} \qquad \lambda_j^i = \frac{1}{\delta^i}\, \exp[-\beta t_j] \left[\frac{1}{16\,\eta_i}\right]^{3/2}.$$

Let now $G_1\big(\epsilon, \beta, Z(\beta t_0)\big)$ be the sum over $0 \le s_i \le 1/\log\beta$ of the positive random variables on the right of (4.5),

$$G_1\big(\epsilon, \beta, Z(\beta t_0)\big) = C_d\, h\, \tilde\epsilon^{\,d-1} \sum_{0 \le i \le \frac{1}{h \log\beta}} \left[\exp\left(-\frac{C}{s_i^2}\right) + g(s_i, \|Z(\beta t_0)\|)\right].$$

By removing the conditioning on (4.5), applying (4.6) and adding over i, we conclude that

$$\begin{aligned} &\sum_{0 \le i \le \frac{1}{h \log\beta}} P\big[\mathcal{A}_i^1 \cap \|Y_i - y_i - \gamma_i(Z_i - z_i)\| \le \eta_i,\, t_0 \le T\big] \\ &\quad \le E\Big[G_1\big(\epsilon, \beta, Z(\beta t_0)\big)\Big] \le C(d, \sigma, T)\, \epsilon^{d-1} \exp\big[-C(\log\beta)^{9/8}\big] \end{aligned} \tag{4.7}$$

On the other hand, it follows from (4.3) that

$$P\left[\mathcal{A}_i^1 \cap \|Y_i - y_i - \gamma_i(Z_i - z_i)\| > \eta_i \,|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right] \\ \leq C_d \left(1 + \|Z(\beta t_0)\|\right) h\, \tilde{\epsilon}^{\,d-1}\, \frac{1}{\alpha_i^{d/2}}\, \exp\left[-\frac{C}{s_i\sqrt{\log\beta}}\right],$$

$0 \leq i \leq 1/(h\log\beta)$. Define the random variable

$$G_2(\epsilon, \beta, Z(\beta t_0)) \\ = \sum_{0\leq i\leq \frac{1}{h\log\beta}} P\left[\mathcal{A}_i^1 \cap \|Y_i - y_i - \gamma_i(Z_i - z_i)\| > \eta_i \,|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right] \\ \leq C_d \left(1 + \|Z(\beta t_0)\|\right) \tilde{\epsilon}^{\,d-1}\, \frac{1}{(\log\beta)^2}$$

Split this last quantity according to whether $\|Z(\beta t_0)\| \leq \log\beta$ or $\|Z(\beta t_0)\| > \log\beta$ and take expectations applying Schwartz inequality to the integral over the latter set, to get

$$E\big[G_2(\epsilon, \beta, Z(\beta t_0))\big] \\ \leq C_d\, \epsilon^{d-1}\, \frac{\sqrt{\beta}}{(\log\beta)^2}\left[1 + E\left[\sup_{0\leq t\leq T}\|Z(\beta t)\|^2\right]^{\frac{1}{2}} P\left[\sup_{0\leq t\leq T}\|Z(\beta t)\| \geq \log\beta\right]^{\frac{1}{2}}\right].$$

Computations similar to those yielding (4.6) allow us to conclude that

$$E\big[G_2(\epsilon, \beta, Z(\beta t_0))\big] = o\left(\sqrt{\beta}\epsilon^{d-1}\right) \tag{4.8}$$

It remains to treat the case $s_i > 1/\log\beta$. We first take the limit $\tilde{\epsilon} \to 0$. From (4.3) we have

$$P\big[\mathcal{A}_i^1 \,|\, \{t_0,\, Z(\beta t_0),\, y_0\}\big] = h\,\tilde{\epsilon}^{\,d-1}\,\omega_{d-1}\int_{\mathbb{R}^d}\|z\|\, p_i(z_i, z)\, p_i(y_i, 0\,|z_i, z)\, dz \\ + h^2\,\tilde{\epsilon}^{\,d-1}\,\varepsilon_i(\beta, t_0, Z(\beta t_0), y_0),$$

where the variables $\{\varepsilon_i\}$ can be bounded uniformly in i by an integrable random variable $C(\beta, t_0, Z(\beta t_0))$ which can be chosen independently of ϵ.

Now add over the set of i's such that $1/\log\beta \leq s_i \leq (\beta/\log\beta)(T - t_0)$. Since the size h of each interval in the partition goes to 0 as $\tilde{\epsilon}$ does, independently of β, we may replace the sum by a Riemann integral. This yields

$$\sum_{\frac{1}{h\,\log\beta}<i} P\big[\mathcal{A}_i^1 \,|\, \{t_0,\, Z(\beta t_0),\, y_0\}\big] \\ = \tilde{\epsilon}^{\,d-1}\,\omega_{d-1}\int_{\frac{1}{\log\beta}}^{\frac{\beta}{\log\beta}(T-t_0)}\left[\int_{\mathbb{R}^d}\|z\|\, p_s(z_s, z)\, p_s(y_s, 0\,|z_s, z)\, dz\right] ds \\ + h\,\tilde{\epsilon}^{\,d-1}\, e(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)), \tag{4.9}$$

if a_s, b_s, c_s, d_s, z_s, y_s and $p_s(z_s,z)$, $p_s(y_s,0\,|z_s,z)$ are defined by substituting the argument s_i by s in the corresponding definitions,

$$\left|e(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0))\right| \le C(\epsilon,\beta,Z(\beta t_0)).$$

Once more, it is necessary to treat the case when $\|Z(\beta t_0)\|$ achieves very large values separately. In this case the threshold is given by $\log\beta$.

We first look at the typical situation $\|Z(\beta t_0)\| \le \log\beta$. A brief consideration of the values a_s, b_s, c_s and d_s in the range of values that s can achieve suggests performing the change of variables $u = s/\log\beta$. Dominated convergence then implies that

$$\begin{aligned}
&1_{\{\|Z(\beta t_0)\|\le\log\beta\}}\,\tilde{\epsilon}^{\,d-1}\,\omega_{d-1}\int_{\frac{1}{\log\beta}}^{\frac{\beta}{\log\beta}(T-t_0)}\left[\int_{\mathbb{R}^d}\|z\|\,p_s(z_s,z)\,p_s(y_s,0\,|z_s,z)\,dz\right]ds\\
&= \tilde{\epsilon}^{\,d-1}\,\omega_{d-1}\,1_{\{\|Z(\beta t_0)\|\le\log\beta\}}\,\log\beta\\
&\qquad\times\int_{\frac{1}{\log^2\beta}}^{\frac{\beta}{\log^2\beta}(T-t_0)}\left[\int_{\mathbb{R}^d}\|z\|\,p_s(z_s,z)\,p_s(y_s,0\,|z_s,z)\,dz\right]du\\
&= c_d\,\frac{1}{\sigma}\,\tilde{\epsilon}^{\,d-1}\,\log\beta\,1_{\left\{\|Z(\beta t_0)\|\le\log\beta\right\}} + G_3(T,t_0,\epsilon,\beta,\|Z(\beta t_0)\|)\,,
\end{aligned} \tag{4.10}$$

where in the last integral s is a function of the variable of integration u, $s = s(u) = u\log\beta$. The constant c_d is given by

$$c_d = \frac{1}{\sqrt{2}}\,\omega_{d-1}\left(\int_0^\infty p_s^1(0,y)\,ds\right)\left(\int_{\mathbb{R}^d}\|z\|p_1^1(0,z)\,dz\right),$$

y any unitary vector $\|y\|=1$, and $G_3(T,t_0,\epsilon,\beta,\|Z(\beta t_0)\|)$ is a random variable with negligible mean,

$$E\left[G_3(T,t_0,\epsilon,\beta,\|Z(\beta t_0)\|)\right] = o\left(\epsilon^{d-1}\sqrt{\beta}\right). \tag{4.11}$$

Finally, let

$$\begin{aligned}
G_4\big(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0)\big) = {}& C(\sigma)\,\tilde{\epsilon}^{\,d-1}\,\frac{\beta\,T}{\log\beta}\,\|Z(\beta t_0)\|\,1_{\{\|Z(\beta t_0)\|>\log\beta\}}\\
&+ h\,\tilde{\epsilon}^{\,d-1}e(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0)),
\end{aligned}$$

$C(\sigma)$ chosen so that

$$\begin{aligned}
&\tilde{\epsilon}^{\,d-1}\,\omega_{d-1}\,1_{\{\|Z(\beta t_0)\|>\log\beta\}}\,\log\beta\\
&\qquad\times\int_{\frac{1}{\log^2\beta}}^{\frac{\beta}{\log^2\beta}(T-t_0)}\left[\int_{\mathbb{R}^d}\|z\|\,p_s(z_s,z)\,p_s(y_s,0\,|z_s,z)\,dz\right]du\\
&\quad + h\,\tilde{\epsilon}^{\,d-1}e(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0))\\
&\qquad\qquad \le G_4\big(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0)\big).
\end{aligned}$$

It follows from the bounds on $P\big[\|Z(\beta t_0)\| > \log\beta\big]$ that

$$E\Big[G_4\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big)\Big] = o(\epsilon^{d-1}). \tag{4.12}$$

The result is now a consequence of (4.2) and then (4.7-12), if we define

$$G\big(h, \tilde\epsilon, \beta, \|Z(\beta t_0)\|\big) = G_1 + G_2 + G_3 + G_4.$$

$\square$

In order to obtain an upper bound to the probability that the process $\tilde Y$ ever reaches the ball of radius $\tilde\epsilon$ centered at the origin, it remains to estimate the probability of the events of type $\mathcal{A}_i^2$.

Lemma 3.

$$\sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P\Big[\mathcal{A}_i^2 \,\Big|\, \{t_0,\, Z(\beta t_0),\, y_0\}\Big]$$
$$\le C(d,\sigma)\,\frac{\tilde\epsilon^d}{h}\log\beta + G'\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big),$$

G' a random variable with

$$E\Big[\big|G'(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\big|\Big] = o\left[\frac{\epsilon^d}{h}\frac{\beta}{\log\beta}\right] = o\big[\sqrt{\beta}\epsilon^{d-1}\big].$$

Proof. We have

$$P\left[\mathcal{A}_i^2 \,\Big|\, \{t_0, Z(\beta t_0), y_0\}\right] \le P\Big[\|Y_i\| \le \tilde\epsilon \,\big|\, \{t_0, Z(\beta t_0), y_0\}\Big]$$
$$+ P\Big[\|Y_i + hZ_i\| < \tilde\epsilon \,\big|\, \{t_0, Z(\beta t_0), y_0\}\Big].$$

The first probability on the right equals an integral on $\mathbb{R}^{2d}$ equal to the one in (4.3), except that the y–vector is integrated over the d–dimensional ball of radius $\tilde\epsilon$ centered at the origin. Adding over i, this yields

$$\sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(t-t_0)\rceil} P\Big[\|Y_i\| \le \tilde\epsilon \,\big|\, \{t_0, Z(\beta t_0), y_0\}\Big]$$
$$\le C(d,\sigma)\,\frac{\tilde\epsilon^d}{h}\log\beta + G_1'(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))$$

where G_1' is a random variable that satisfies

$$E\Big[\big|G_1'(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\big|\Big] = o\left[\frac{\epsilon^d}{h}\frac{\beta}{\log\beta}\right].$$

In order to obtain a similar expression for the sum of the remaining terms

$$P\Big[\|Y_i + hZ_i\| < \tilde\epsilon \,\big|\, \{t_0, Z(\beta t_0), y_0\}\Big],$$

it is useful to consider the Gaussian vector $(Z_i, Y_i + hZ_i)$. Computations analogous to the ones leading to the bound on $P\big[\|Y_i\| \le \tilde\epsilon \,\big|\, \{t_0, Z(\beta t_0), y_0\}\big]$ then show that

$$\sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(t-t_0)\rceil} P\Big[\|Y_i + hZ_i\| < \tilde\epsilon \,\Big|\, \{t_0, Z(\beta t_0), y_0\}\Big]$$
$$\le C(d,\sigma)\,\frac{\tilde\epsilon^d}{h}\log\beta + G_2'(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)),$$

as well, where G_2' is such that

$$E\Big[\big|G_2'(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\big|\Big] = o\left[\frac{\epsilon^d}{h}\frac{\beta}{\log\beta}\right].$$

Take

$$G'(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)) = G_1' + G_2'$$

to finish the proof. □

Corollary 1. *There exists a random variable*

$$H(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)),$$
$$E\Big[\big|\,H\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big)\big|\Big] = o\big[\sqrt{\beta}\epsilon^{d-1}\big],$$

such that

$$P\Big[\inf_{0\le t\le\frac{\beta}{\log\beta}(T-t_0)} \|\tilde Y(t)\| \le \tilde\epsilon \,\Big|\, \{t_0,\, Z(\beta t_0),\, y_0\}\Big]$$
$$\le \sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P\left[\mathcal{A}_i \,\middle|\, \{t_0, y_0, Z(\beta t_0)\}\right]$$
$$= c_d\,\frac{1}{\sigma}\,\tilde\epsilon^{\,d-1}\,\log\beta \; + H\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big),$$

c_d as in the statement of Lemma 2.

Proof. Immediate from the proof of Lemma 2, and Lemma 3. □

4.2 A lower bound

We can write

$$
\begin{aligned}
P\Big[&\inf_{0\le t\le \frac{\beta}{\log\beta}(T-t_0)} \|\tilde{Y}(t)\| \le \tilde{\epsilon}\,\Big|\,\{t_0,\, Z(\beta t_0),\, y_0\}\Big] \\
&= P\left[\bigcup_{0\le t\le \frac{\beta}{\log\beta}(T-t_0)} \mathcal{A}_i \,\middle|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right] \\
&= \sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P\left[\mathcal{A}_i\big|\,\{t_0,\, Z(\beta t_0),\, y_0\}\right] \\
&\quad - \sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P\left[\mathcal{A}_i \cap \bigcup_{i<j<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} \mathcal{A}_j \,\middle|\, \{t_0,\, Z(\beta t_0),\, y_0\}\right],
\end{aligned}
\tag{4.13}
$$

where we recall that the event $\mathcal{A}_i$ is given by

$$
\mathcal{A}_i = \left\{\inf_{s_i\le t<s_{i+1}} \|\tilde{Y}(t)\| \le \tilde{\epsilon}\right\}.
$$

We will now show

Lemma 4.

$$
\begin{aligned}
\sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P&\left[\mathcal{A}_i \cap \bigcup_{i<j<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} \mathcal{A}_j \,\middle|\, \{t_0, Z(\beta t_0), y_0\}\right] \\
&\le C(d,\sigma)\,\frac{\log\beta}{\beta}\,\tilde{\epsilon}^{\,d-1} + R(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\,,
\end{aligned}
$$

with $R(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))$ an integrable random variable such that

$$
E\Big[R(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\Big] = o\left[\frac{\epsilon^{d-1}}{\sqrt{\beta}}\right],
$$

Proof. Suppose that the event $\mathcal{A}_i$ occurs, and let $\tilde{Y}_i$ and Z_i be the values achieved by $\tilde{Y}$ and $Z(\beta t_0 + \cdot\,\log\beta)$ at time s_i. Let $j > i$. Due to the definition of $\tilde{Y}(\cdot)$, a necessary condition for $\mathcal{A}_j$ to happen is that the component of the vector $Z_k = Z(\beta t_0 + s_k \log\beta)$ in the direction Z_i becomes smaller than

$\delta = \tilde{\epsilon}^{1+\mu}$, for some $i \leq k \leq j$. In particular,

$$
\begin{aligned}
&\bigcup_{i<j\leq\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} \mathcal{A}_i \cap \mathcal{A}_j \\
&\subseteq \left(\mathcal{A}_i \,\cap\, \left\{ \|Z_i\| \leq \frac{1}{\beta} \right\} \right) \cup \left(\mathcal{A}_i \,\cap\, \left\{ \|Z_i\| \geq \frac{1}{\beta} \right\} \cap \bigcup_{i<k\leq k(\tilde{\epsilon})} \left\{ \langle Z_k, Z_i \rangle \leq \delta \right\} \right) \\
&\qquad \cup \left(\mathcal{A}_i \,\cap \bigcup_{k(\tilde{\epsilon})<k<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} \mathcal{A}_k \right)
\end{aligned} \tag{4.14}
$$

with

$$
k(\tilde{\epsilon}) = \left\lceil \frac{\tilde{\epsilon}^{\frac{1}{4}}}{h} \right\rceil .
$$

It follows from (4.3) and analogous expressions for the probability that $\tilde{Y}_i$ belongs to one of the caps of the cylinder $\mathcal{C}_i$ while $\|Z_i\| \leq 1/\beta$, that there exists a random variable $R_1(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))$ such that

$$
\begin{aligned}
\sum_{0\leq i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} & P\left[\mathcal{A}_i \cap \left\{ \|Z_i\| \leq \frac{1}{\beta} \right\} \middle| \{t_0, Z(\beta t_0), y_0\} \right] \\
&\leq C(d,\sigma)\, \frac{\log\beta}{\beta}\, \tilde{\epsilon}^{\,d-1} + R_1(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)),
\end{aligned} \tag{4.15}
$$

$$
E\Big[R_1(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)) \Big] = o\left(\frac{\epsilon^{d-1}}{\sqrt{\beta}} \right).
$$

Also, conditional to the values of $(Z_i, \tilde{Y}_i)$, the random vector $(Z_k, \tilde{Y}_k)$ is Gaussian with mean

$$
\left(e^{-(s_k-s_i)\log\beta}\, Z_i,\, \tilde{Y}_i + \frac{1-e^{-(s_k-s_i)\log\beta}}{\log\beta}\, Z_i \right)
$$

and covariance matrix

$$
\Gamma_k^i = \begin{pmatrix} a_k^i\, I_d & b_k^i\, I_d \\ b_k^i\, I_d & c_k^i\, I_d \end{pmatrix}
$$

where

$$
\begin{aligned}
a_k^i &= \frac{\sigma^2}{2} \big[1 - e^{-2(s_k-s_i)\log\beta} \big] \\
b_k^i &= \frac{\sigma^2}{2}\, \frac{1}{\log\beta} \big[1 - 2e^{-(s_k-s_i)\log\beta} + e^{-2(s_k-s_i)\log\beta} \big] \\
&= \frac{\sigma^2}{2}\, \frac{1}{\log\beta} \big[1 - e^{-(s_k-s_i)\log\beta} \big]^2
\end{aligned}
$$

and

$$c_k^i = \sigma^2 \Big[\frac{s_k - s_i}{\log\beta} - \frac{2}{(\log\beta)^2} \big(1 - e^{-(s_k - s_i)\log\beta}\big) + \frac{1}{2(\log\beta)^2} \big(1 - e^{-2(s_k - s_i)\log\beta}\big) \Big].$$

Computations similar to those leading to (4.3) then yield

$$P\left[\mathcal{A}_k \,\middle|\, \{\mathcal{A}_i,\, Z_i,\, \tilde{Y}_i\}\right] \le C(d,\sigma)\,(\log\beta)^{\frac{d-1}{2}}\,\tilde{\epsilon}^{\frac{11}{4}}\,h + R_2^{ik}(\epsilon,\beta,Z_i,\tilde{Y}_i)$$

whenever $k \ge k(\tilde{\epsilon})$, where the random variables R_2^{ik} satisfy

$$E\Big[R_2^{ik}(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0))\Big] = o\left[\epsilon^{d-1+\frac{11}{4}}\,h^2\,\beta^2\,(\log\beta)^{\frac{d-1}{2}}\right].$$

Adding over k and integrating over the set $\mathcal{A}_i$ and the vector $(Zi, \tilde{Y}_i)$, we conclude that

$$P\left[\mathcal{A}_i \cap \bigcup_{k(\tilde{\epsilon}) \le k \le \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)} \mathcal{A}_k \,\middle|\, \{t_0, Z(\beta t_0), y_0\}\right] \le C(d,\sigma,T)\,\beta\,(\log\beta)^{\frac{d-1}{2}}\,\tilde{\epsilon}^{\,d-1+\frac{11}{4}}\,h + R_2^i(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0)) \quad (4.16)$$

with

$$E\left[R_2^i(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0))\right] = o\left[\epsilon^{d-1+\frac{11}{4}}\,h\,\beta^3\,(\log\beta)^{\frac{d-1}{2}}\right].$$

It remains to estimate

$$\mathcal{A}_i \cap \left\{\|Z_i\| \ge \frac{1}{\beta}\right\} \cap \bigcup_{i<k\le k(\tilde{\epsilon})} \left\{\langle Z_k, Z_i\rangle \le \delta\right\}.$$

Let i and $Z_i, \|Z_i\| > 1/\beta$, be given. Then

$$Z(\beta t_0 + s\log\beta) = e^{-(s-s_i)\log\beta} Z_i + \sigma\, e^{-(s-s_i)\log\beta} \int_0^{(s-s_i)\log\beta} e^u\, d\tilde{W}(u),$$

$\tilde{W}(\cdot)$ a standard d–dimensional Brownian motion. Now, the Wiener measure is invariant under multiplication by unitary matrices, so by performing an orthogonal change of coordinates if necessary, we may assume that $Z_i/\|Z_i\|$ is the first vector in the canonical basis:

$$Z_i = \|Z_i\|\, e_1\,.$$

We then have

$$P\left[\bigcup_{i<k\le k(\tilde\epsilon)}\Big\{\big\langle Z_k, Z_i\big\rangle \le \delta\Big\}\,\Bigg|\,\Big\{\mathcal{A}_i, Z_i, \|Z_i\| \ge \frac{1}{\beta}\Big\}\right]$$
$$\le P\left[\sup_{s_i\le s\le s_i+\tilde\epsilon^{\,1/4}} \int_0^{(s-s_i)\log\beta} e^u\, dW_1(u) \,\ge\, \frac{1}{2\beta}\right],$$

provided ϵ is sufficiently small. Here $W_1(\cdot)$ is a 1–dimensional Brownian motion. By Doob's inequality,

$$P\left[\bigcup_{i<k\le k(\tilde\epsilon)}\Big\{\big\langle Z_k, Z_i\big\rangle \le \delta\Big\}\,\Bigg|\,\Big\{\mathcal{A}_i, Z_i, \|Z_i\| \ge \frac{1}{\beta}\Big\}\right] \le 8\,\beta^2\,\log\beta\;\tilde\epsilon^{\frac{1}{4}} \tag{4.17}$$

Decomposition (4.14) and estimates (4.15), (4.16) and (4.17) imply that

$$\sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P\left[\mathcal{A}_i \cap \bigcup_{i<j<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} \mathcal{A}_j \,\Bigg|\, \{t_0, Z(\beta t_0), y_0\}\right]$$
$$\le C(d,\sigma)\,\frac{\log\beta}{\beta}\,\tilde\epsilon^{\,d-1} + R(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\,,$$

where $R(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)) = R_1 + R_2$ satisfies

$$E\Big[R(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))\Big] = o\left[\frac{\epsilon^{d-1}}{\sqrt{\beta}}\right],$$

as claimed. $\square$

Corollary 2. *Let c_d be the constant from the statement of Lemma 2. Then*

$$P\Big[\inf_{0\le t\le\frac{\beta}{\log\beta}(T-t_0)} \|\tilde Y(t)\| \le \tilde\epsilon\,\Big|\,\{t_0,\, Z(\beta t_0),\, y_0\}\Big]$$
$$\ge c_d\,\frac{1}{\sigma}\,\tilde\epsilon^{\,d-1}\,\log\beta\; + \tilde H\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big),$$

where the random variable $\tilde H(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0))$ satisfies

$$E\Big[\big|\,\tilde H\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big)\big|\Big] = o\big[\sqrt{\beta}\epsilon^{d-1}\big].$$

Proof. The result follows from (4.13), Lemma 4 and the identity

$$\sum_{0\le i<\lceil \frac{1}{h}\frac{\beta}{\log\beta}(T-t_0)\rceil} P\left[\mathcal{A}_i \,\middle|\, \{t_0, y_0, Z(\beta t_0)\}\right]$$
$$= c_d\,\frac{1}{\sigma}\,\tilde\epsilon^{\,d-1}\,\log\beta\; + \mathcal{H}\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big),$$
$$E\Big[\big|\,\mathcal{H}\big(T, t_0, \epsilon, \beta, y_0, Z(\beta t_0)\big)\big|\Big] = o\big[\sqrt{\beta}\epsilon^{d-1}\big]\,.$$

$\square$

Corollaries 1 and 2 clearly imply that

$$P\Big[\inf_{0\le t\le \frac{\beta}{\log\beta}(T-t_0)} \|\tilde{Y}(t)\| \le \tilde{\epsilon}\,\Big|\, \{t_0,\, Z(\beta t_0),\, y_0\}\Big]$$
$$= c_d\,\frac{1}{\sigma}\,\tilde{\epsilon}^{\,d-1}\,\log\beta\ + C\big(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0)\big)\,,$$

where the random variable C has negligible mean,

$$E\left[\left|C\big(T,t_0,\epsilon,\beta,y_0,Z(\beta t_0)\big)\right|\right] = o\left[\sqrt{\beta}\epsilon^{d-1}\right].$$

5 First passage time estimate

We start by studying the accuracy of the piecewise linear approximation $\tilde{Y}(\cdot)$ to the process $Y(\cdot)$.

Lemma 5. *Let $\delta = \tilde{\epsilon}^{\,1+\mu}$, $\mu > 0$ as in the definition of h. Then*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{\,d-1}}\,P\left[\sup_{0\le u\le \frac{\beta}{\log\beta}(T-t_0)} \|Y(u)-\tilde{Y}(u)\| \ge \delta,\ t_0\le T\right] = 0.$$

Proof. It is clear from the definition of $\tilde{Y}$ that

$$P\left[\sup_{0\le u\le \frac{\beta}{\log\beta}(T-t_0)} \|Y(u)-\tilde{Y}(u)\| \ge \delta,\ t_0\le T\right]$$
$$\le P\left[\sup_{\substack{\beta t_0\le s<t\le\beta T\\ 0\le t-s\le h\log\beta}} \|Z(t)-Z(s)\| \ge \frac{\delta}{h},\ t_0\le T\right].$$

We now work on this last expression. Due to the SDE satisfied by $Z(\cdot)$, it will be enough to control

$$P\left[\sup_{\substack{\beta t_0\le s<t\le\beta T\\ 0\le t-s\le h\log\beta}} \int_s^t \|Z(u)\|\,du \ge \frac{\delta}{h},\ t_0\le T\right] \tag{5.1}$$

$$\text{and}\qquad P\left[\sup_{\substack{\beta t_0\le s<t\le\beta T\\ 0\le t-s\le h\log\beta}} \|W(t)-W(s)\| \ge \frac{\delta}{\sigma h},\ t_0\le T\right], \tag{5.2}$$

where we recall that $W(\cdot)$ is a standard d–dimensional Brownian motion.

We start with (5.1). As a first observation, note that it is bounded by

$$P\left[\sup_{0\le s\le\beta T} \|Z(s)\| \ge \frac{\delta}{h^2\log\beta}\right].$$

It follows from the representation $Z(t) = e^{-t}Z(0) + \sigma e^{-t}\int_0^t e^s\,dW(s)$ that

$$P\left[\sup_{0\le s\le \beta T}\|Z(s)\| \ge \frac{\delta}{h^2\log\beta}\right] \le P\left[\sup_{0\le s\le \beta T}\Big\| Z(0) + \sigma\int_0^s e^u\,dW(u)\Big\| \ge \frac{\delta}{h^2\log\beta}\right].$$

The rest of the argument is similar to the one applied in the proof of Proposition 1 to derive (3.2), only simpler, as in the present case it is not necessary to consider a partition of $[0,\beta T]$. We describe it very briefly now. The first term $Z(0)$ on the right is a $N(0,\sigma^2/2\,I_d)$–random vector. In order to control the second term, the problem is first reduced to that of a 1–dimensional Brownian motion, $W_1(\cdot)$ say, by considering the coordinates of $W(\cdot)$. We then apply Doob's inequality to the Brownian exponential martingale

$$\exp\left\{\lambda\int_0^t e^s\,dW_1(s) - \frac{\lambda^2}{2}\int_0^t e^{2s}\,ds\right\},$$

with $\lambda = C(d,\sigma)\,e^{-\beta T}$ to conclude that

$$P\left[\sup_{\substack{\beta t_0\le s<t\le\beta T\\ 0\le t-s\le h\log\beta}}\int_s^t\|Z(u)\|\,du \ge \frac{\delta}{h},\ t_0\le T\right] \le C(d,\sigma)\,\exp\left\{-\frac{\delta}{h^2}\frac{e^{-\beta T}}{\log\beta}\right\} \tag{5.3}$$

We turn to (5.2). We apply Garsia, Rodemich and Rumsey inequality (see [13], Chapter 2) to each path $W(\cdot)$ with a choice of increasing functions $\Psi(u) = u^\alpha$, for

$$\alpha\ >\ 4\vee\left[\frac{2}{\mu}\Big(d+\frac{1}{3}+2\mu\Big)\right],$$

and $p(u) = \sqrt{u}$. This leads to

$$P\left[\sup_{\substack{\beta t_0\le s<t\le\beta T\\ 0\le t-s\le h\log\beta}}\|W(t)-W(s)\| \ge \frac{\delta}{\sigma h},\ t_0\le T\right] \le P\left[C\,(\log\beta)^{\frac12-\frac2\alpha}\,\Gamma^{\frac1\alpha} \ge \frac{\delta}{h^{\frac32-\frac2\alpha}}\right],$$

for some positive constant C that depends only on the diffusion coefficient σ and the parameter α. Here Γ is given by

$$\int_0^{\beta T}\int_0^{\beta T}\Psi\Big(\frac{\|W(t)-W(s)\|}{\sqrt{t-s}}\Big)\,ds\,dt = \int_0^{\beta T}\int_0^{\beta T}\frac{\|W(t)-W(s)\|^\alpha}{|t-s|^{\alpha/2}}\,ds\,dt.$$

By Tchebyshev's inequality, we get

$$
\begin{aligned}
P\left[\sup_{\substack{\beta t_0\le s<t\le\beta T\\ 0\le t-s\le h\log\beta}}\|W(t)-W(s)\|\ge\frac{\delta}{\sigma h},\ t_0\le T\right]&\\
&\le C(\alpha,\sigma,d)\,\frac{h^{\frac{3}{2}\alpha-2}}{\delta^\alpha}\,(\log\beta)^{\frac{\alpha}{2}-2}\,E\,[\Gamma]\\
&\le C(\alpha,\sigma,d)\,\frac{h^{\frac{3}{2}\alpha-2}}{\delta^\alpha}\,(\log\beta)^{\frac{\alpha}{2}-2}\,(\beta T)^2 \quad (5.4)
\end{aligned}
$$

The lemma follows from (5.3), (5.4) and the choice of α. □

Proof of Proposition 2. Note that

$$\delta=\tilde\epsilon^{1+\mu}=o(\tilde\epsilon)\,.$$

The chain of inclusions

$$
\begin{aligned}
\big\{\|Y-\tilde Y\|\le\delta,\ \|\tilde Y\|\le\tilde\epsilon-\delta\big\}&\\
&\subseteq\ \big\{\|Y\|\le\tilde\epsilon\big\}\\
&\subseteq\ \big\{\|\tilde Y\|\le\tilde\epsilon+\delta,\ \|\tilde Y-Y\|\le\delta\big\}\cup\big\{\|Y-\tilde Y\|>\delta\big\}
\end{aligned}
$$

and Lemma 5 then imply

$$
\begin{aligned}
P\left[\inf_{0\le u\le\frac{\beta}{\log\beta}(T-t_0)}\|Y(u)\|\le\tilde\epsilon,\ t_0\le T\right]&\\
&=P\left[\inf_{0\le u\le\frac{\beta}{\log\beta}(T-t_0)}\|\tilde Y(u)\|\le\tilde\epsilon,\ t_0\le T\right]+o\left[\epsilon^{d-1}\right],
\end{aligned}
$$

where we recall that

$$\big\{t_0\le T\big\}=\left\{\inf_{0\le t\le T}\|X(t)\|\le\frac{\log\beta}{\sqrt\beta}\right\}.$$

We thus obtain

$$
\begin{aligned}
&P\left[\inf_{0\le t\le T}\|X(t)\|\le\epsilon\right]\\
&\quad=P\left[\inf_{0\le t\le T}\|X(t)\|\le\frac{\log\beta}{\sqrt\beta}\quad\text{and}\quad\inf_{0\le u\le\frac{\beta}{\log\beta}(T-t_0)}\|Y(u)\|\le\tilde\epsilon\right]\\
&\quad=P\left[\inf_{0\le t\le T}\|X(t)\|\le\frac{\log\beta}{\sqrt\beta}\quad\text{and}\quad\inf_{0\le u\le\frac{\beta}{\log\beta}(T-t_0)}\|\tilde Y(u)\|\le\tilde\epsilon\right]+o\left[\epsilon^{d-1}\right]\\
&\quad=E\left[1_{\left\{\inf_{0\le t\le T}\|X(t)\|\le\frac{\log\beta}{\sqrt\beta}\right\}}\,E\left[\inf_{0\le u\le\frac{\beta}{\log\beta}}\|\tilde Y(u)\|\le\tilde\epsilon\,\middle|\,\{t_0,Z(\beta t_0),y_0\}\right]\right]\\
&\qquad\qquad+o\left[\epsilon^{d-1}\right]\\
&\quad=c_d\,\frac{1}{\sigma}\,\tilde\epsilon^{d-1}\,\log\beta\,P\left[\inf_{0\le t\le T}\|X(t)\|\le\frac{\log\beta}{\sqrt\beta}\right]+o\left[\sqrt\beta\epsilon^{d-1}\right].
\end{aligned}
$$

The last line is a consequence of the observation following the proof of Corollary 2 in the previous section, c_d the constant appearing in the statement of Lemma 2. By Proposition 1, we conclude that

$$P\left[\inf_{0\le t\le T}\|X(t)\|\le\epsilon\right]=C_d\,\sqrt{\beta}\,\epsilon^{d-1}\,\sigma\,\int_0^T p_s^\sigma(0,x_0)\,ds+o\left[\sqrt{\beta}\epsilon^{d-1}\right]$$

with

$$C_d=\frac{1}{\sqrt{2}}\,(\frac{d}{2}-1)\,\omega_{d-1}\,\omega_d\left(\int_0^\infty p_s^1(0,y)\,ds\right)\left(\int_{\mathbb{R}^d}\|z\|\,p_1^1(0,z)\,dz\right),$$

as claimed. In this last formula y is any vector with $\|y\|=1$, and ω_d denotes the volume of the d–dimensional unit sphere. □

References

[1] Armendáriz, I. (2006) Brownian coagulation and a version of Smoluchowski's equation on the circle. *Preprint.*

[2] Billingsley, P. (1968). *Convergence of probability measures.* J. Wiley & Sons, Inc., New York–London–Sidney.

[3] Deaconu, M. and Fournier, N. (2002). Probabilistic approach of some discrete and continuous coagulation equations with diffusion. *Stochastic Process. Appl.* **101**, 83–111.

[4] Freidlin, M. (2004). Some remarks on the Smoluchowski-Kramers approximation. *J. Stat. Phys.* **117**, 3/4, 617–634.

[5] Großkinsky, S., Klingenberg, C. and Oelschläger, K. (2003) A rigorous derivation of Smoluchowski's equation in the moderate limit. *Stochastic. Anal. Appl.*, **22**, 1, 113-141.

[6] Hammond, A. and Rezakhanlou, F. (2006) Kinetic limit for a system of coagulating Brownian particles. *Preprint.*

[7] Lang, R. and Xuan-Xanh, N. (1980). Smoluchowski's theory of coagulation in colloids holds rigorously in the Boltzmann–Grad limit. *Z. Wahrscheinlichkeitstheorie verw. Geb.* **54**, 227-280.

[8] Le Gall, J–F. (1986). Sur la saucisse de Wiener et les points multiples du mouvement Brownien. *Ann.Probab.* **14**, 2, 1219–1244.

[9] Norris, J. R. (2006). Coagulation of Brownian particles. *Preprint.*

[10] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion.* 3rd. ed. Springer.

[11] van Smoluchowski, M. (1916). Drei Vorträge über Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen. *Physik. Z.* **17**, 557–585.

[12] Spitzer, F. (1958). Some theorems concerning 2–dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87**, 187–197.

[13] Stroock D. W. and Varadhan, S. R. S. (1997). *Multidimentional Diffusion Processes.* 2nd. ed. Springer.