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On the Logarithmic Derivative of the ζ-Determinant operators with boundary [*][†]

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Abstract

A formula for the derivative of the logarithm of the ζ-determinant of the quotient of two elliptic differential operators with the same boundary condition, acting between the fibers of a vector bundle over a n-dimensional manifold M with boundary X, is here presented.

1 Introduction

Given a trace class operator A acting on a Hilbert space, the Fredholm determinant of the operator $L = I - A$ is defined by

$$det_1 L = \prod_{j=1}^{\infty}(1 - \lambda_j(A)), \tag{1}$$

where I is the identity operator and the numbers $\lambda_j(A)$ are the eigenvalues of A, repeated the times indicated by their corresponding multiplicities.

It is a very known fact the necessity of this concept in various areas of mathematics, as differential geometry [4], and those of physics, for instance in the construction of quantum theories by means of functional integration (see for instance [12], [5], [6], [3]), in where the calculus of determinants of quotients of some elliptic differential operators recovers a special interest.

In [4] R. Forman has studied some Fredholm determinant properties of L and the quotient of regularization of the determinants of two differential operators D_0 and D_1 by the Riemann ζ-function method, when $L = D_0 D_1^{-1} = I - A$ and A belongs to the trace class operators. This type of determinant regularization procedure is called the ζ-determinant regularization and is denoted by Det_ζ.

In general, for an operator L acting on a Hilbert space H the notions of Fredholm determinant and the ζ-determinant might have no sense. Nevertheless, when the Fredholm determinant of some operator having the form $L = L_1 L_0^{-1}$ exists it can be extended, under some hypotheses, by the ratio of the regularized ζ-determinants of the involved operators. Then, in several occasions, the interest is focalized on the quotient of the determinants of the operators instead of each determinant individually. On this line the works [5] and [6] fit in perfectly. It is shown in such papers that the quotient between the ζ-determinants of two elliptic operators $A + \epsilon A_1$ and A, defined on a compact differential manifold without boundary, is given by

$$\frac{Det_\zeta(A + \epsilon A_1)}{Det_\zeta(A)} = \exp\left\{\epsilon \left.\frac{d}{ds}\right|_{s=0}\left[s.Tr\big(A^{-s-1}A_1\big)\right] + O(\epsilon^2)\right\}, \tag{2}$$

where A is pseudodifferential of positive order and A_1 is a differential operator with order$(A_1) <$ order(A). Another version about the derivative of the logarithm of the ζ-determinant with respect to a parameter is presented in [4] where it is established that

$$
\begin{aligned}
\frac{d}{dt} \log Det_\zeta L_{tB} &= \frac{d}{ds} Tr \left[s. \left(\frac{d}{dt} L_{tB} \right) . L_{tB}^{-s-1} \right] \Bigg|_{s=0} \\
&= \frac{d}{dt} \log det_1 \left(L_{tB}.L_{0B}^{-1} \right),
\end{aligned}
\tag{3}
$$

for a quotient of two elliptic differential operators with boundary conditions belonging to a family of operators L_{tB}, all with identical principal symbol and the same elliptic boundary condition B. The strong hypothesis required by R. Forman is that $\left(\dfrac{d}{dt} L_{tB} \right) L_{tB}^{-1}$, the logarithmic derivative of L_{tB}, is a trace class operator for all t. In some recent papers it is possible to find many results concerning the regularization of functional determinants applied to particular models, see for instance [3] , [5], [6] and [8].

The aim of this paper is to extend 3 to the quotient of two classical elliptic differential operators with elliptic boundary conditions without assuming the trace class condition about the logarithmic derivative of L_{tB}. In [1] a very close result is obtained for pseudodifferential operators without boundary conditions on a compact manifold.

In the next section we introduce the main results without proof. In section three we establish the notation and recall the definitions and some properties about the ζ-determinant regularization method and the Fredholm determinant. The last section is devoted to the proofs.

2　Main results

The principal statements about the logarithm of the ζ-determinant of the quotient of two elliptic differential operators with the same boundary elliptic condition are presented.

Theorem 2.1.
Let Ω be an open subset of the complex plane and let $z(t) : [0,1] \longrightarrow \Omega$ be a differentiable curve. Over a compact, n-dimensional, differential manifold M with boundary X define

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the z-analytic family $\{L_z\}_{z\in\Omega}$ of elliptic, invertible, differential operators, having order $m > 0$. Let B be the same elliptic boundary condition for each L_z. Let denote with L_t the elliptic problem $(L_{z(t)}, B)$, for all $t \in [0, 1]$.

It is supposed that all the operators of the family have the same principal symbol, which has a cone of minimum growth rays.

Then, for all $t \in [0, 1]$ it is satisfied

$$\frac{d}{dt} \ln Det_\zeta L_t = \left.\frac{d}{ds}\right|_{s=0} Tr\left[s.\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right], \tag{4}$$

being the r.h.s. of this equality the finite part at $s = 0$ of the meromorphic extension of $Tr\left[\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]$.

Recall that the finite part at $s = 0$ of a function with a simple pole there is to subtract the term with the pole at $s = 0$ to the function, and then to take the limit as $s \to 0$.

Remark 2.2. *Following the result given by theorem 2 in [10] and the steps of its proof it can be shown that, taking $A = \left(\frac{d}{dt}L_t\right).L_t^{-1}$, the kernel of the composition operator $A.L_t^{-s}$ can be extended to a meromorphic function of s with only simple poles located at the same values of the simple poles of the kernel of the operator L_t^{-s}.*

Next, the corresponding integrated version will be enunciated. In order to deduce the first corollary it is enough to take the exponential function after integrating from 0 to t_o in equation (4) of the previous theorem.

Corollary 2.3. *(Integrated version)*
Under the hypotheses of theorem 2.1, it is true that

$$\frac{Det_\zeta L_{t_o}}{Det_\zeta L_0} = \exp\left\{\int_0^{t_o} \left.\frac{d}{ds}\right|_{s=0}\left\{s.Tr\left[\frac{d}{dt}(L_t).L_t^{-s-1}\right]\right\} dt\right\}.$$

Corollary 2.4. *(Logarithmic derivative for the trace class case)*
Under all the hypotheses of theorem 2.1, if besides $\left(\frac{d}{dt}L_t\right).L_t^{-1}$ is a trace class operator for all t, it is valid that

$$\begin{aligned}
\frac{d}{dt} \log Det_\zeta L_t &= \left.\frac{d}{ds} Tr\left[s.\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]\right|_{s=0} \\
&= \frac{d}{dt} \log det_1\left(L_t.L_0^{-1}\right), \tag{5}
\end{aligned}$$

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and also

$$\frac{Det_\zeta L_{t_o}}{Det_\zeta L_0} = det_1\left(L_{t_o}.L_0^{-1}\right).$$

Remark 2.5. *It should be noted that the previous corollary is just one of the results established in [4].*

3 Setting down the frame

Let M be a differential manifold equipped by a measure μ. The space of all the complex valued functions defined over M having derivatives of every order will be denoted by

$$\mathcal{C}^\infty(M) = \{f : M \longrightarrow \mathbf{C} \ \ f \text{ is infinitely differentiable}\}. \tag{6}$$

In general, H will be understood a Hilbert space and the set of all the linear and continuous operators $T : H \longrightarrow H$ will be denoted $\mathcal{L}(H)$. In particular, the Hilbert space of the square integrable functions $f : M \longrightarrow \mathbf{C}$ will be denoted by $H = L^2(M)$.

3.1 Trace class operators and Fredholm determinant

A compact operator A defined on a Hilbert space H is called to be a trace class operator if

$$Tr(|A|) = \sum_{j=1}^\infty \mu_j(A) < \infty, \tag{7}$$

where $\mu_j(A)$, the singular values of A, are the eigenvalues of $|A| = \sqrt{A^*A}$. The set of the trace class operators on H form an ideal denoted $\mathcal{J}_1$. If I denotes the identity operator on H, the Fredholm determinant of $L = I - A$ was defined by (1) as

$$det_1 L = \prod_{j=1}^\infty (1 - \lambda_j),$$

where $\{\lambda_j(A)\}_j$ denotes the proper values of A when A is a trace class operator. Of course, its trace is given by

$$Tr(A) = \sum_{j=1}^\infty \lambda_j(A) < \infty.$$

The expression (7) defines a norm on $\mathcal{J}_1$, called the trace norm and denoted $\|A(z)\|_1 = Tr(|A|)$.

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In this paper, it will be applied the integral representation of the Fredholm determinant (c.f. [7])

$$det_1(I - A) = \exp\left\{-\int_\gamma Tr\left[A\left(1 - zA\right)^{-1}\right] dz\right\}, \tag{8}$$

with $\gamma : [0,1] \longrightarrow \mathbf{C}$ a continuous path such that $\gamma(0) = 0$, $\gamma(1) = 1$ and that the operator $(1 - zA)^{-1}$ exists and is bounded for all z in γ.

Some properties connected with the differentiability of the Fredholm determinants are recalled now. The corresponding proofs can be found in [2].

Lemma 3.1.

Let $A(z) : G \longrightarrow \mathcal{J}_1$ a holomorphic application over an open subset G of $\mathbf{C}$ taking values on the ideal $\mathcal{J}_1$ of the trace class operators equipped with the norm of $\mathcal{L}(H)$. Let us suppose that the trace norm $\|A(z)\|_1$ of $A(z)$ is bounded over each compact subset of G.
Then, the funtion $det_1(I - A(z)) : G \longrightarrow \mathbf{C}$ is holomorphic.

Lemma 3.2.

Under the hypotheses of lemma 3.1 we have

- *the derivative of $A(z)$ is a trace class operator for all $z \in G$;*

- *the funtion $Tr(A(z))$ is holomorphic on G, and*

- $\dfrac{d}{dz}\left[Tr(A(z))\right] = Tr\left[\dfrac{d}{dz}A(z)\right].$

Remark 3.3. *Since $\mathcal{J}_1$ is not a closed subspace of $\mathcal{L}(H)$ in the norm of the bounded operators, the first statement is not evident at all.*

Lemma 3.4.

Under the hypotheses of lemma 3.1 it results

$$\frac{d}{dz}ln(det_1(I - A(z)) = -Tr\left[(I - A(z))^{-1}\frac{d}{dz}(A(z))\right].$$

Remark 3.5. *Let us notice the very close connection between this last lemma and the formula (8) given in [7].*

3.2 ζ-determinant

Let us treat the case of an elliptic differential operator L of order $m > 0$ in a smooth bounded domain M with boundary conditions B given on ∂M. Let L_B be the operator L with domain $\{u \in C^\infty : Bu = 0 \text{ on } \partial M\}$ and we denote also by L_B its closure in $L^2(M)$. Since L_B is an unbounded operator, it is clear that the product of the eigenvalues is divergent. So, the way to define the determinant of L_B as the product of its eigenvalues is not convenient, unless the operator $I - L_B$ is a trace class operator. Therefore, in order to define $DetL_B$ we appel to the notion of the trace $Tr(L_B^{-s})$, called the generalized Riemann ζ-function associated to L_B and denoted by $\zeta(L_B, s)$. For this issue, we need to define the complex powers L_B^{-s}, then we choose the boundary condition B such the operator $L_B - \lambda$ is invertible for all λ belonging to an appropriate sector $\Gamma \subset \mathbf{C}$ (for example, if the system of operators (L, B) is elliptic and satifies the Agmon's condition, see for instance [9] and [11]).

Then, for a given complex number s, one of the way to define the operator L_B^{-s} is ([9] and [11])

$$L_B^{-s} = \frac{i}{2\pi} \int_\Gamma \lambda^{-s} (L_B - \lambda I)^{-1} d\lambda, \quad \text{if} \quad \mathbf{R}e(s) > 0$$

$$\tag{9}$$

$$L_B^{-s} = L_B^k . L_B^{-(k+s)}, \quad \text{if} \quad -k < \mathbf{R}e(s) \leq -(k-1) \leq 0,$$

with $k \geq 1$ an integer number and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ the path on the complex plane, where for some angle θ each path is defined by

$$\Gamma_1 = \{te^{i\theta}\} \quad , \quad \text{varying } t \text{ from } \infty \text{ to } \epsilon > 0,$$

$$\Gamma_2 = \{|\lambda| = \epsilon\} \quad \text{clockwise oriented, and} \tag{10}$$

$$\Gamma_3 = \{te^{i\theta}\} \quad , \quad \text{varying } t \text{ from } \epsilon \text{ to } \infty,$$

assuming that there exists a cone of directions around the ray $\arg \lambda = \theta$ in such a way that no eigenvalue of L belongs to the cone. In [9] and [10] it was proved that the function $Tr(L_B^{-s})$ is holomorphic in a half-plane and that admits a meromorphic extension to the whole complex s-plane, being analytic at $s = 0$. As a consequence, one of the possible notions of the regularized determinant of the operator L_B is known as the of the

generalized Riemann ζ-function and that, in what follows, it will be denoted $Det_\zeta L_B$. Therefore,

$$Det_\zeta L_B = \exp\left\{ -\frac{d}{ds}\Big|_{s=0} \zeta(L_B, s) \right\}. \tag{11}$$

4   Proofs

Proofs of theorem 2.1

The proof of this theorem is essentially the same as the analogous theorem in [1]. Under the hypotheses of the theorem, the complex powers of L_t are given by 9.

Let $k > \dfrac{n}{m}$ be an integer and $s \in \mathbf{C}$ such that $\mathbf{R}e(s) \geq k$. According to [9], [10], and [13], L_t^{-s} is a trace class operator and its kernel is continuous on the diagonal of M. Since the complex powers depend analytically on the parameter s, from lemma 3.2 it follows for $\mathbf{R}e(s) > k$ that

$$
\begin{aligned}
\frac{d}{dt} Tr\left(L_t^{-s}\right) &= \frac{d}{dt} Tr\left[L_t^{k-s}\, L_t^{-k}\right] \\
&= \frac{d}{dt} Tr\left[\frac{i}{2\pi} \int_\Gamma \lambda^{k-s}(L_t - \lambda)^{-1} L_t^{-k}\, d\lambda \right] \\
&= Tr\left\{ \frac{i}{2\pi} \int_\Gamma \lambda^{k-s}\left[-(L_t - \lambda)^{-1} \frac{d}{dt}(L_t)\,(L_t - \lambda)^{-1} L_t^{-k} + \right.\right. \\
&\quad \left.\left. + (L_t - \lambda)^{-1} \left(\sum_{j=1}^k L_t^{-j+1} \frac{d}{dt}(L_t^{-1}) L_t^{-k+j} \right) \right] d\lambda \right\} \\
&= -\frac{i}{2\pi} \int_\Gamma \lambda^{k-s} Tr\left[(L_t - \lambda)^{-1} \frac{d}{dt}(L_t)\,(L_t - \lambda)^{-1} L_t^{-k}\, d\lambda \right] + \\
&\quad + \frac{i}{2\pi} \sum_{j=1}^k \int_\Gamma \lambda^{k-s} Tr\left[-(L_t - \lambda)^{-1} L_t^{-j+1} L_t^{-1} \frac{d}{dt}(L_t) L_t^{-1} L_t^{-k+j} \right] d\lambda.
\end{aligned}
$$

By the cyclic property of the trace it can be written as

$$\frac{d}{dt}Tr(L_t^{-s}) = -\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}Tr\left((L_t-\lambda)^{-2}\left(\frac{d}{dt}L_t\right)L_t^{-k}\right)d\lambda -$$

$$-\frac{i}{2\pi}\sum_{j=1}^{k}\int_\Gamma \lambda^{k-s}Tr\left[(L_t-\lambda)^{-1}\left(\frac{d}{dt}L_t\right)L_t^{-k-1}\right]d\lambda$$

$$= Tr\left[-\frac{i}{2\pi}\int_\Gamma \lambda^{k-s}(L_t-\lambda)^{-2}d\lambda\left(\frac{d}{dt}L_t\right)L_t^{-k}\right] - kTr\left[L_t^{k-s}\left(\frac{d}{dt}L_t\right)L_t^{-k-1}\right].$$

Integrating by parts and taking into account that $\mathbf{R}e(s) > k$, we have

$$\frac{d}{dt}Tr(L_t^{-s}) = Tr\left[(k-s)\left(\frac{d}{dt}L_t\right)L_t^{-s-1} - k\left(\frac{d}{dt}L_t\right)L_t^{-s-1}\right]$$

$$= Tr\left[-s\left(\frac{d}{dt}L_t\right)L_t^{-s-1}\right]$$

$$= (-s).Tr\left[\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]. \tag{12}$$

As a function of s the r.h.s. of (12) has a meromorphic extension to the whole complex plane ([9], [10], [11] and [13]) with only simple poles possibly localized at $s = n - \dfrac{j}{m}$, for $j = 1, 2, \dots$. In particular, at $s = 0$ such extension is analytical.

Eventually, in virtue of definition of ζ-determinant given by formula (11) and expression (12), it is clear that

$$\frac{d}{dt}\ln Det_\zeta L_t = \frac{d}{dt}\left\{-\frac{d}{ds}\Big|_{s=0}Tr(L_t^{-s})\right\}$$

$$= -\frac{d}{ds}\Big|_{s=0}\left\{\frac{d}{dt}Tr(L_t^{-s})\right\}$$

$$= \frac{d}{ds}\Big|_{s=0}\left\{s.Tr\left[\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]\right\}.$$

$\Diamond$

Proof of corollary 2.4

In fact, from the integral representation (8) of the Fredholm determinant det_1 it results that

$$\frac{d}{ds}\Big|_{s=0}\left\{s.Tr\left[\left(\frac{d}{dt}L_t\right).L_t^{-s-1}\right]\right\} = Tr\left[\left(\frac{d}{dt}L_t\right).L_t^{-1}\right] = \frac{d}{dt}\ln det_1\left(L_t.L_0^{-1}\right).$$

The conclusion follows straightforward after integrating the last equality from 0 to t_o and taking the exponential function. $\diamond$

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