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“Linearization of Functions”

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LINEARIZATION OF FUNCTIONS

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INTRODUCTION.

In this paper we present a general construction linearizing functions with values in locally convex spaces. Loosely speaking, given a space of scalar-valued functions $\mathcal{F}(U)$ on the set U , what we construct is a space $\mathcal{F}_*(U)$ and a function $e : U \rightarrow \mathcal{F}_*(U)$ of the same *type* as those of $\mathcal{F}(U)$ factoring all functions of $\mathcal{F}(U)$ and identifying each with a continuous linear form $L : \mathcal{F}_*(U) \rightarrow \mathbb{C}$; that is

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{C} \\ e \downarrow & \nearrow L & \\ \mathcal{F}_*(U) & & \end{array}$$

In fact we seek to identify $\mathcal{F}(U)$ algebraically and topologically with the dual of $\mathcal{F}_*(U)$, and indeed, the space of F -valued functions $\mathcal{F}(U, F)$ with the continuous linear operators $L(\mathcal{F}_*(U), F)$. As will be shown below, the existence of such a ‘linearization’ is strictly stronger than the mere existence of a predual. Although we concentrate primarily on some classes of holomorphic functions, our approach is general enough to produce linearizations of other function spaces as well.

Linearizations of this type have been obtained for spaces of continuous homogeneous polynomials by R. Ryan [12] through the use of symmetric tensor products. In the holomorphic setting, linearizations have been constructed for holomorphic functions by P. Mazet [7] and S. Dineen [4], for bounded holomorphic functions by J. Mujica [8], and for holomorphic functions of bounded type by P. Galindo, D. García and M. Maestre [5] (see also [1], [10]). These constructions involve some insertion into the dual of $\mathcal{F}(U)$, and the use of the Dixmier-Nejman theorem [11]. We have sought to clarify the subject through an abstract approach that captures the essential ingredients of linearizations. Though we generalize all the constructions mentioned above, we feel that ours is more akin to Ryan’s approach and can be viewed as a generalization of tensor products.

The paper is organized as follows. In section 1 we construct a space X , which when topologized and completed will be $\mathcal{F}_*(U)$. We also define the map e mentioned above. The dual pairing $\langle X, \mathcal{F}(U) \rangle$ is discussed, as well as algebraic aspects of the duality. Two somewhat independent issues arise: the topology on X' , and the topology on X . These questions are addressed in sections 2 and 3 respectively.

These constitute the core of the paper. In section 2, we discuss two strong topologies on X' and obtain a topological (as well as algebraic) isomorphism $X' = \mathcal{F}(U)$ in many cases. From section 3 on, we restrict our attention to spaces of functions which are continuous. We complete our definition of $\mathcal{F}_*(U)$, discuss the universal factorization property, and characterize $\mathcal{F}_*(U)$. Section 4 is devoted to considering the case of vector-valued functions. These results and those at the beginning of section 5 enable us to present a number of concrete examples of $\mathcal{F}_*(U)$. These include preduals of the spaces (to be defined below) $P^k(E)$, $P_I^k(E)$, $H^\infty(U)$, $H_b(U)$, $H_I(B_E^\circ)$, $H(U)$ and $C(U)$, of homogeneous polynomials, integral homogeneous polynomials, holomorphic bounded functions, holomorphic functions of bounded type, integral holomorphic functions, holomorphic functions, and continuous functions.

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1. THE CONSTRUCTION OF X AND THE MAP e .

Throughout, E and F will be locally convex spaces. U is, in general, simply a set, though for some of our results we will require it to be a topological space, or a subset of E . These situations will be clear from the context. Most of our results are valid for real or complex spaces, but for simplicity, we will consider only complex spaces in our exposition. Fix a space $\mathcal{F}(U)$ of scalar-valued functions defined over U . We proceed now to construct X and e .

We consider first the vector space $\mathbb{C}^{(U)}$ of finitely supported families of U -indexed complex numbers. A typical element will be denoted by $s = \sum_{x \in U} a_x e_x$, with $e_x(y) = \delta_{xy}$. Note that the sum is finite. For any given $f \in \mathcal{F}(U)$, we define the seminorm

$$p_f(s) = \left| \sum_{x \in U} a_x f(x) \right|.$$

Now

$$\mathcal{N} = \{s \in \mathbb{C}^{(U)} : p_f(s) = 0 \text{ for all } f \in \mathcal{F}(U)\}$$

is a subspace of $\mathbb{C}^{(U)}$, and we define X as the quotient

$$X = \mathbb{C}^{(U)} / \mathcal{N}.$$

We will continue to denote the class of e_x by e_x , and the class of s by $s = \sum_{x \in U} a_x e_x$. Note that now this way of writing the class of s need not be unique. However, the seminorms on X , originated by $\left| \sum_{x \in U} a_x f(x) \right|$ (which we continue to denote p_f) are well-defined and

provide a Hausdorff locally convex space structure for X . We will later change the topology on X , but this will do for now. Also, define

$$e : U \longrightarrow X, \quad \text{by } e(x) = e_x.$$

A few comments are in order regarding X and the notation $s = \sum_{x \in U} a_x e_x$. The space \mathcal{N} may be very large, and indeed, in the case $\mathcal{F}(E) = P^k(E)$ the quotient involves considerable collapsing. However, if $U \subset E$ and the space $\mathcal{F}(U)$ contains sufficiently many polynomials of high degree, as is the case with all spaces of holomorphic functions, $\mathcal{N} = 0$ and thus X is simply $\mathbb{C}^{(U)}$ with the corresponding seminorms. We prove this in the following proposition.

Proposition 1.1. *Suppose $\mathcal{F}(U)$ contains the restrictions of the polynomials γ^k , where γ ranges over all continuous linear forms on E , and $k = 0, 1, 2, \dots$. Then the space \mathcal{N} is zero.*

Proof: Take $s = \sum_{x \in U} a_x e_x \in \mathcal{N}$, and write x_1, \dots, x_n for those elements of U in the support of s , a_i for a_{x_i} , and $\widehat{x}(\gamma) = \gamma(x)$. We have

$$\sum_{i=1}^n a_i \widehat{x}_i^0 = 0, \quad \sum_{i=1}^n a_i \widehat{x}_i = 0, \quad \sum_{i=1}^n a_i \widehat{x}_i^2 = 0, \quad \dots, \quad \sum_{i=1}^n a_i \widehat{x}_i^{n-1} = 0;$$

the equalities above as functions over the dual of E . Written in matrix form,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \widehat{x}_1 & \widehat{x}_2 & \cdots & \widehat{x}_n \\ \widehat{x}_1^2 & \widehat{x}_2^2 & \cdots & \widehat{x}_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x}_1^{n-1} & \widehat{x}_2^{n-1} & \cdots & \widehat{x}_n^{n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This system will have only the trivial solution if its determinant is non-zero. But this Vandermonde determinant is $\prod_{i < j} (\widehat{x}_i - \widehat{x}_j)$. Since there are continuous linear functionals γ over E with $\gamma(x_i) \neq \gamma(x_j)$ for $i \neq j$, then a_1, \dots, a_n must all be zero. Thus $s = 0$. ■

Now, it is clear that a function $f \in \mathcal{F}(U)$ factors through e in the following way

$$\begin{array}{ccc} U & \xrightarrow{f} & \mathbb{C} \\ e \downarrow & \nearrow_{L_f} & \\ X & & \end{array}$$

where we define $L_f(s) = \sum_x a_x f(x)$ if $s = \sum_x a_x e_x$. Such L_f is well-defined, linear, and continuous:

$$|L_f(s)| = \left| \sum_x a_x f(x) \right| = p_f(s).$$

The map $\mathcal{F}(U) \longrightarrow X'$ defined by $f \mapsto L_f$ is an algebraic isomorphism. Indeed, the pairing $\langle X, \mathcal{F}(U) \rangle$ given by $\langle s, f \rangle = L_f(s)$ is a dual pairing, so algebraically we have $X' = \mathcal{F}(U)$ [6]. This duality is the basis for the topological considerations in the next sections.

We have introduced a topology on X by means of the seminorms p_f , where f ranges over all functions of $\mathcal{F}(U)$. We shall call this topology τ . There are of course many topologies on X which are compatible with τ (i.e., topologies with the same continuous linear functionals). For any one of them the dual space X' is identified algebraically with $\mathcal{F}(U)$.

We now face two issues which are relatively independent: the topology to be introduced in X' , and the topology to be introduced in X . On X' we would like a topology which will coincide with $\mathcal{F}(U)$'s original topology in a number of important examples, obtaining $X' = \mathcal{F}(U)$ both algebraically and topologically. On the other hand, we wish to introduce on X the finest possible topology for which the map $e : U \longrightarrow X$ will retain properties common to all $f \in \mathcal{F}(U)$.

2. THE DUAL PAIRING $\langle X, \mathcal{F}(U) \rangle$ AND TOPOLOGIES ON X' .

Recall that all topologies on X which are compatible with τ give rise to the same bounded subsets of X [6]. We shall refer to these bounded sets as ' \mathcal{F} -bounded', and also call ' \mathcal{F} -bounded' the $\sigma(X'', \mathcal{F}(U))$ -bounded subsets of X'' . We shall also refer to \mathcal{F} -bounded subsets of U ; by these we mean subsets of U on which all functions of $\mathcal{F}(U)$ are bounded. This notion is usually called 'bounding' in the case of spaces of holomorphic functions.

In many situations the space $\mathcal{F}(U)$ is the 'strong dual' of X , in other words, that the topology on $\mathcal{F}(U)$ coincides with the topology of uniform convergence on \mathcal{F} -bounded subsets of X when functions f are identified with the linear functionals L_f . In what follows we will denote by (X', β) the strong dual of X . We will denote by (X', β') the space $\mathcal{F}(U)$ with the topology of uniform convergence on \mathcal{F} -bounded subsets of X'' . Recall [6] that this is a barrelled space. Also, given an \mathcal{F} -bounded set B , we denote by q_B the seminorm on X'

$$q_B(L) = \sup_{s \in B} |L(s)|.$$

Note that if $\mathcal{F}(U)$ is barrelled, its topology is the strong topology $\beta(\mathcal{F}(U), \mathcal{F}(U)')$ [6], which we have denoted β' . Thus, if $\mathcal{F}(U)$ is barrelled, we have $\mathcal{F}(U) = (X', \beta')$. Another case in which something can be said is when the topology on $\mathcal{F}(U)$ is given by uniform convergence on \mathcal{F} -bounded subsets of U . We see this in the theorem below. We will need the following definition.

Definition 2.1. *We shall say that U has the $BB\mathcal{F}$ -property if for each \mathcal{F} -bounded subset B of X , there is an \mathcal{F} -bounded subset A of U and*

an $r > 0$ such that

$$B \subset r \operatorname{coe}(e(A)).$$

Theorem 2.2. *Suppose $\mathcal{F}(U)$ is barrelled, and its topology is that of uniform convergence on \mathcal{F} -bounded subsets of U . Then*

- i) $\mathcal{F}(U) = (X', \beta)$, and
- ii) U has the $BB\mathcal{F}$ -property.

Proof: i) Let A be an \mathcal{F} -bounded subset of U , and set $B = e(A)$. B is of course \mathcal{F} -bounded in X , and

$$\sup_{x \in A} |f(x)| = \sup_{e_x \in B} |L_f(e_x)| = q_B(L_f) \quad \text{for all } f \in \mathcal{F}(U).$$

Thus we have that $(X', \beta) \longrightarrow \mathcal{F}(U)$ is continuous. But since $\mathcal{F}(U)$ is barrelled, the map is also open.

ii) We know that $\mathcal{F}(U) \longrightarrow (X', \beta)$ with $f \mapsto L_f$ is a topological isomorphism. Thus for each \mathcal{F} -bounded B in X , there is an $r > 0$ and an \mathcal{F} -bounded A in U with

$$q_B(L_f) \leq r \sup_{x \in A} |f(x)| \quad \text{for all } f \in \mathcal{F}(U).$$

Thus for each $s \in B$ we have

$$|L(s)| \leq q_B(L) \leq r \sup_{e_x \in e(A)} |L(e_x)| \quad \text{for all } L \in X'.$$

so $s \in r \operatorname{coe}(e(A))$ for all $s \in B$, and $B \subset r \operatorname{coe}(e(A))$. ■

3. TOPOLOGIES ON X , AND THE PREDUAL $\mathcal{F}_*(U)$.

We now face our second problem. We want a particular topology for our predual, and we want it to be as strong as possible. From this section on, we restrict ourselves to classes $\mathcal{F}(U)$ of functions which are continuous. Thus U is a topological space. Our main consideration here is that the map $e : U \longrightarrow X$ be continuous, but with the strongest compatible topology that will accomplish this, to produce a ‘small’ complete predual. Accordingly, we choose to complete the definition of $\mathcal{F}_*(U)$ as follows.

Definition 3.1. *Consider on X the strongest locally convex topology compatible with τ for which the map $e : U \longrightarrow X$ is continuous, and denote X with this topology by (X, α) . We take $\mathcal{F}_*(U)$ to be the completion of (X, α)*

$$\mathcal{F}_*(U) = \widehat{(X, \alpha)}.$$

We still denote by e the continuous map $U \longrightarrow X \longrightarrow \mathcal{F}_*(U)$.

In quite general situations the map e will turn out to retain the common properties of $f \in \mathcal{F}(U)$: the definition of the topology on $\mathcal{F}_*(U)$ forces e to inherit properties of $f \in \mathcal{F}(U)$. Thus e will be, ‘bounded’, ‘ k -homogeneous polynomial’, or ‘holomorphic’, if all $f \in \mathcal{F}(U)$ share these properties.

Proposition 3.2. *If all functions $f \in \mathcal{F}(U)$ are bounded (resp. holomorphic; k -homogeneous polynomials), then $e : U \rightarrow \mathcal{F}_*(U)$ is bounded (resp. holomorphic; a k -homogeneous polynomial).*

Proof: ‘bounded’: $p_f(e(U)) = \{|f(x)| : x \in U\}$ is bounded for all $f \in \mathcal{F}(U)$. Hence $e(U)$ is bounded for the τ topology, but by the Banach-Mackey theorem, all topologies compatible with τ produce the same bounded sets, thus $e(U)$ is bounded in $\mathcal{F}_*(U)$.

‘holomorphic’: Given any continuous linear functional L on $\mathcal{F}_*(U)$, $L \circ e \in \mathcal{F}(U)$. Thus e is weakly holomorphic. Since e is continuous and $\mathcal{F}_*(U)$ complete, e is holomorphic.

‘ k -homogeneous polynomial’: For every $f \in \mathcal{F}(U)$, $\lambda \in \mathbb{C}$, and $x \in U$,

$$p_f(e(\lambda x) - \lambda^k e(x)) = |f(\lambda x) - \lambda^k f(x)| = 0.$$

Thus $e(\lambda x) = \lambda^k e(x)$, so e is k -homogeneous. Since it is also holomorphic, it is a k -homogeneous polynomial. ■

The predual $\mathcal{F}_*(U)$ which we have constructed, together with the map e , are characterized by the following universal property.

Theorem 3.3. *$\mathcal{F}_*(U)$ is a complete locally convex space, the map $e : U \rightarrow \mathcal{F}_*(U)$ is continuous, and each $f \in \mathcal{F}(U)$ factors linearly through e :*

$$\begin{array}{ccc} U & \rightarrow & \mathbb{C} \\ e \downarrow & \nearrow L & \\ \mathcal{F}_*(U) & & \end{array}$$

identifying $\mathcal{F}_*(U)'$ with $\mathcal{F}(U)$ algebraically. Furthermore, if Y is another space which has these properties, there is a continuous injection $\mathcal{F}_*(U) \rightarrow Y$ with dense image making the following diagram commute:

$$\begin{array}{ccc} U & & \\ \downarrow & \searrow & \\ \mathcal{F}_*(U) & \rightarrow & Y \end{array}$$

These properties characterize the pair $\mathcal{F}_*(U), e$ up to isomorphism.

Proof: The first assertions are clear. Suppose that Y is a complete algebraic predual of $\mathcal{F}(U)$, and each $f \in \mathcal{F}(U)$ factors linearly through a continuous \tilde{e}

$$\begin{array}{ccc} U & \rightarrow & \mathbb{C} \\ \tilde{e} \downarrow & \nearrow T_f & \\ Y & & \end{array}$$

Define a linear $\Lambda : (X, \alpha) \rightarrow Y$ by setting $\Lambda(e_x) = \tilde{e}(x)$. Λ is one-to-one: for each $f \in \mathcal{F}(U)$ we have $T_f \circ \Lambda = L_f$ so if $\Lambda(s) = 0$, then $L_f(s) = 0$ for all f and hence $s = 0$.

Also, Λ is continuous: let q be a continuous seminorm on Y . Then $p = q \circ \Lambda$ is a seminorm on X , and we need only show that it is α -continuous. The seminorm p is ‘compatible’ with τ , because if $L : X \rightarrow \mathbb{C}$ is p -continuous, then $T = L \circ \Lambda^{-1} : \text{Im} \Lambda \rightarrow \mathbb{C}$ is q -continuous because $T(\Lambda(s)) = L(s) \leq cp(s) = cq(\Lambda(s))$ for all s . Now extend T to $\overline{T} \in \mathcal{F}(U)$ by Hahn-Banach, and we have $\overline{T} = T_f$. Thus $L = T_f \circ \Lambda = L_f$. But e is p -continuous. Indeed, let $x_i \rightarrow x$ be a convergent net in U . Then

$$p(e_{x_i} - e_x) = q(\Lambda(e_{x_i}) - \Lambda(e_x)) = q(\tilde{e}(x_i) - \tilde{e}(x)) \rightarrow 0,$$

so p is a seminorm of the α topology, and Λ is continuous.

Note also that $\text{Im} \Lambda = [\tilde{e}(U)]$ is dense. If not, there would be a non-zero $T_f \in Y' = \mathcal{F}(U)$ such that $L_f = T_f \circ \Lambda = 0$, which is absurd because both T_f and L_f identify with the same $f \in \mathcal{F}(U)$. Now, taking completions, we have

$$\widehat{\Lambda} : \mathcal{F}_*(U) \rightarrow Y.$$

$\widehat{\Lambda}$ is continuous, has dense image, and is still one-to-one: note that $T_f \circ \widehat{\Lambda} = \widehat{L}_f$, so if $\widehat{\Lambda}(s) = 0$, $\widehat{L}_f(s) = 0$ for all $f \in \mathcal{F}(U)$, and hence $s = 0$.

Finally, if (Z, \bar{e}) has all these properties, we have

$$\begin{array}{ccc} U & & \\ e \downarrow & \searrow \bar{e} & \\ \mathcal{F}_*(U) & \leftrightarrow & Z \end{array}$$

where the horizontal arrows are continuous, one-to-one, and their compositions coincide with the identities on dense subspaces. Thus $Z = \mathcal{F}_*(U)$. ■

4. THE CASE OF VECTOR-VALUED FUNCTIONS.

We consider now the vector-valued case. We wish to show that our predual $\mathcal{F}_*(U)$ factors not only scalar-valued functions, but also the vector-valued functions in $\mathcal{F}(U, F)$. This will serve, in some cases, to verify that our construction $\mathcal{F}_*(U)$ coincides with preduals constructed by others. It is of course not always clear exactly what the F -valued functions in the ‘class of $\mathcal{F}(U)$ ’ are, but in instances such as ‘holomorphic’ or ‘continuous’ it is. In any case, the functions that we will effectively be able to linearize through $\mathcal{F}_*(U)$ are those in the following class, so we begin by defining them.

Definition 4.1. *Given a class $\mathcal{F}(U)$ of continuous scalar-valued functions, and a locally convex space F , we will say that $g : U \rightarrow F$ is weakly in \mathcal{F} if g is continuous and for every $\gamma \in F'$, $\gamma \circ g \in \mathcal{F}(U)$. We denote by $\omega\mathcal{F}(U, F)$ the space of all such functions.*

Observe that $e : U \rightarrow \mathcal{F}_*(U)$ is weakly in \mathcal{F} , and that for any $T \in L(\mathcal{F}_*(U), F)$, $T \circ e \in \omega\mathcal{F}(U, F)$. Note also that for some classes of functions (continuous, holomorphic, polynomials), ‘weakly in \mathcal{F} ’ is no restriction, i.e.: $\omega C(U, F) = C(U, F)$, $\omega H(U, F) = H(U, F)$, and $\omega P^k(U, F) = P^k(U, F)$. For others, such as the class of integral polynomials, such equalities do not hold.

Theorem 4.2. *Each function in $\omega\mathcal{F}(U, F)$ factors linearly through e :*

$$\begin{array}{ccc} U & \rightarrow & F \\ e \downarrow & \nearrow_L & \\ \mathcal{F}_*(U) & & \end{array}$$

identifying $L(\mathcal{F}_*(U), F)$ with $\omega\mathcal{F}(U, F)$ algebraically.

Proof: The map $L(\mathcal{F}_*(U), F) \rightarrow \omega\mathcal{F}(U, F)$ defined by $T \mapsto T \circ e$ is an algebraic isomorphism: Clearly the map $T \mapsto T \circ e$ is linear. We wish to see that for each $g \in \omega\mathcal{F}(U, F)$ there is a unique $L_g \in L(\mathcal{F}_*(U), F)$ such that $L_g \circ e = g$. Consider, then, $g \in \omega\mathcal{F}(U, F)$. We define $L_g : X_\alpha \rightarrow F$, and then extend to the completion. There is only one way to do this (so uniqueness will be clear):

$$L_g \left(\sum_{x \in U} a_x e_x \right) = \sum_{x \in U} a_x g(x).$$

We need only show that L_g is well defined and continuous. If $\sum_{x \in U} a_x e_x = 0$, for any $\gamma \in F'$,

$$\left| \gamma \left(\sum_{x \in U} a_x g(x) \right) \right| = \left| \sum_{x \in U} a_x (\gamma \circ g)(x) \right| = p_{\gamma \circ g} \left(\sum_{x \in U} a_x e_x \right) = p_{\gamma \circ g}(0) = 0.$$

Thus $\sum_{x \in U} a_x g(x) = 0$.

Now the continuity. Take a continuous seminorm q of F . We show that $p = q \circ L_g$ is a continuous seminorm in the α topology of X . e is p -continuous: given $\varepsilon > 0$ and $x_0 \in U$, since g is continuous, there is a neighborhood V of x_0 such that $q(g(x) - g(x_0)) < \varepsilon$ for all $x \in V$. Thus for all such x ,

$$p(e_x - e_{x_0}) = q(L_g(e_x) - L_g(e_{x_0})) = q(g(x) - g(x_0)) < \varepsilon.$$

Also, p is ‘compatible’ (each p -continuous linear functional corresponds to an $f \in \mathcal{F}(U)$). Let $L : X \rightarrow \mathbb{C}$ be such a functional. So for all $s \in X$, $|L(s)| \leq cp(s)$. Now define $\gamma : \text{Im} L_g \rightarrow \mathbb{C}$ by $\gamma(L_g(s)) = L(s)$. γ is well defined and continuous: if $L_g(s) = 0$, $p(s) = q(L_g(s)) = 0$, and $L(s) = 0$. The continuity: $|\gamma(L_g(s))| = |L(s)| \leq cp(s) = cq(L_g(s))$. Extend γ to F by Hahn-Banach, and we have $L = L_{\gamma \circ g}$.

Thus p is an α -continuous seminorm, and $q(L_g(s)) = p(s)$, so L_g is continuous. \blacksquare

Corollary 4.3. *If Y is a complete locally convex space and $\tilde{e} : U \longrightarrow Y$ continuous and such that for all complete locally convex spaces F , each $g \in \omega\mathcal{F}(U, F)$ factors through \tilde{e} :*

$$\begin{array}{ccc} U & \xrightarrow{g} & F \\ \tilde{e} \downarrow & \nearrow_{T_g} & \\ Y & & \end{array}$$

identifying $L(Y, F)$ with $\omega\mathcal{F}(U, F)$, then $Y = \mathcal{F}_(U)$.*

Proof: $\tilde{e} \in \omega\mathcal{F}(U, Y)$, so the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\tilde{e}} & Y \\ e \downarrow & \nearrow_{\mathcal{F}_*(U)} & \\ \mathcal{F}_*(U) & & \end{array}$$

where the arrows are $L_{\tilde{e}}$ and T_e . But $L_{\tilde{e}} = \Lambda$, which we have seen in Theorem 3.3 is one-to-one and has dense image. $L_{\tilde{e}} \circ T_e$ is the identity on $[Im\tilde{e}] = Im\Lambda$, which is dense, and $T_e \circ L_{\tilde{e}}$ is the identity on $[Ime]$ which is also dense. Thus $Y = \mathcal{F}_*(U)$. \blacksquare

5. EXAMPLES.

We have constructed, for a large class of function spaces $\mathcal{F}(U)$, a topological vector space $\mathcal{F}_*(U)$ whose strong dual $(\mathcal{F}_*(U)', \beta)$ or $(\mathcal{F}_*(U)', \beta')$ is $\mathcal{F}(U)$. In this section we take a look at particular cases of function spaces $\mathcal{F}(U)$ to see the result of this construction, and comment on some aspects of each case. The results of the preceding section, and those below will allow us to compare our predual $\mathcal{F}_*(U)$ with those constructed for particular cases by R. Ryan [12], S. Dineen [3] [4], J. Mujica [8], P. Galindo, D. Garcıa and M. Maestre [5], and V. Dimant, P. Galindo, M. Maestre and I. Zalduendo [2].

We begin by giving a characterization of the topology α . Consider on $\mathcal{F}(U)'$ the topology (which we will still call α) of uniform convergence on the polars of α -neighborhoods of zero in (X, α) , and denote $\mathcal{F}(U)'$ with this topology by $\mathcal{F}(U)'_\alpha$. In the following proposition, τ_p denotes the topology of pointwise convergence in $\mathcal{F}(U)$, and ‘equicontinuous’ refers to equicontinuity as functions over U .

Proposition 5.1. *The topology α in $\mathcal{F}(U)'_\alpha$ is the topology of uniform convergence on the equicontinuous τ_p -compact disks of $\mathcal{F}(U)$.*

Proof: Call α' the topology on X given by the family of seminorms $p_B(s) = \sup_{f \in B} |L_f(s)|$ where B are equicontinuous τ_p -compact disks of $\mathcal{F}(U)$. It will be enough to see that $\alpha = \alpha'$ on X .

To see $\alpha \leq \alpha'$, take W an α -neighborhood of zero. We check that its polar W° is a τ_p -compact disk and equicontinuous. First, since W is a neighborhood of zero in the Mackey topology, W° is $\sigma(\mathcal{F}(U), X)$ -compact; thus τ_p -compact. Now the equicontinuity. Let $x \in U$ and

$\varepsilon > 0$. $e_x + \varepsilon W$ is a neighborhood of e_x . Since e is continuous, there is a neighborhood V_x of x such that

$$e(V_x) \subset e_x + \varepsilon W.$$

Thus for all $y \in V_x$, $e_y \in e_x + \varepsilon W$ so $e_y - e_x \in \varepsilon W$. Hence for all $f \in W^\circ$, $|f(y) - f(x)| \leq \varepsilon$. We have proved that every α -neighborhood of zero is an α' -neighborhood of zero, so $\alpha \leq \alpha'$.

We now prove $\alpha' \leq \alpha$. Since α is the largest topology compatible with τ making e continuous, it will be enough to check that α' has these two properties. If B is τ_p -compact, for each $s \in X$

$$\sup_{f \in B} |L_f(s)| \leq c_s < \infty$$

and we have $B \subset \{h \in \mathcal{F}(U) : |L_h(s)| \leq c_s \text{ for all } s \in X\}$, which is a $\sigma(\mathcal{F}(U), X)$ -compact disk, so α' is coarser than the Mackey topology. e is α' -continuous: let B be equicontinuous, and $\varepsilon > 0$. There is a neighborhood V_x of x such that $|f(y) - f(x)| < \varepsilon$ for all $y \in V_x$ and $f \in B$, so

$$p_B(e_y - e_x) = \sup_{f \in B} |f(y) - f(x)| < \varepsilon.$$

Thus e is continuous for the α' topology. $\alpha' \leq \alpha$, and both topologies are the same. \blacksquare

In the next lemma, X and Y are arbitrary, though in our use of the lemma they will be as above. Note that we require only X 'contained' in Y , perhaps via a one-to-one map $X \longrightarrow Y$ which need not be continuous.

Lemma 5.2. *Suppose X and Y are locally convex spaces, $X' = Y' = \mathcal{F}$ algebraically, and X a subset of Y . Suppose also that the following condition holds:*

- (*) *for each $y \in Y$ there is a bounded net $(x_i) \subset X$ such that x_i converges to y uniformly on $\sigma(\mathcal{F}, Y)$ -bounded subsets of \mathcal{F} .*

Then given any disk A in \mathcal{F} ; A is $\sigma(\mathcal{F}, Y)$ -compact if and only if it is $\sigma(\mathcal{F}, X)$ -compact.

Proof: \Rightarrow is clear. To see \Leftarrow , take A to be a $\sigma(\mathcal{F}, X)$ -compact disk in \mathcal{F} .

We prove first that it is $\sigma(\mathcal{F}, Y)$ -bounded, and hence $\sigma(\mathcal{F}, Y)$ -precompact. Given $y \in Y$, we want to see that $\sup_{a \in A} |y(a)| < \infty$. Let $(x_i) \subset X$ be a bounded net such that x_i converges to y uniformly on $\sigma(\mathcal{F}, Y)$ -bounded subsets of \mathcal{F} . Since A is a $\sigma(\mathcal{F}, X)$ -compact disk, its polar $A^\circ = \{x \in X : |x(a)| \leq 1 \text{ for all } a \in A\}$ is a zero neighborhood in X . Since (x_i) is bounded, $(x_i) \subset cA^\circ$ for some positive c . Thus

$$|x_i(a)| \leq c \text{ for all } a \in A \text{ and all } i.$$

Since x_i converges to y pointwise, $\sup_{a \in A} |y(a)| \leq c < \infty$.

Now we check that A is $\sigma(\mathcal{F}, Y)$ -complete. Let $(a_j) \subset A$ be a $\sigma(\mathcal{F}, Y)$ -Cauchy net. It is therefore a $\sigma(\mathcal{F}, X)$ -Cauchy net, so there is an $a \in A$ such that $a_j \rightarrow a$ in the $\sigma(\mathcal{F}, X)$ topology. We want to see that this convergence is also in the $\sigma(\mathcal{F}, Y)$ topology. Take $y \in Y$, and $\varepsilon > 0$. Let $(x_i) \subset X$ be a bounded net such that x_i converges to y uniformly on $\sigma(\mathcal{F}, Y)$ -bounded subsets of \mathcal{F} . Since, as we have seen, A is $\sigma(\mathcal{F}, Y)$ -bounded, there is an i_ε such that

$$|x_{i_\varepsilon}(b) - y(b)| < \frac{\varepsilon}{3} \text{ for all } b \in A.$$

Hence

$$|y(a_j) - y(a)| \leq |y(a_j) - x_{i_\varepsilon}(a_j)| + |x_{i_\varepsilon}(a_j) - x_{i_\varepsilon}(a)| + |x_{i_\varepsilon}(a) - y(a)| < \varepsilon$$

for j large enough so that $|x_{i_\varepsilon}(a_j) - x_{i_\varepsilon}(a)| < \frac{\varepsilon}{3}$. Thus, A is $\sigma(\mathcal{F}, Y)$ -compact. ■

If Y is as in Theorem 3.3, define $J_Y : Y \rightarrow \mathcal{F}(U)'$ by $J_Y(y)(f) = T_f(y)$. Also, define $J_X : (X, \alpha) \rightarrow \mathcal{F}(U)'_\alpha$ by $J_X(s)(f) = L_f(s)$. Then the topology induced by $\mathcal{F}(U)'_\alpha$ on $Im J_X$ makes J_X a homeomorphism. We will need the following lemma.

Lemma 5.3. *If $J_Y : Y \rightarrow \mathcal{F}(U)'_\alpha$ is continuous, then $Y = \mathcal{F}_*(U)$.*

Proof: As in Theorem 3.3, we have the continuous one-to-one map $\Lambda : (X, \alpha) \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} & Y & \\ \nearrow & \tilde{e} \uparrow & \searrow^{J_Y} \\ \Lambda \mid & U & \rightarrow \mathcal{F}(U)'_\alpha \\ \searrow & e \downarrow & \nearrow_{J_X} \\ & (X, \alpha) & \end{array}$$

where the horizontal arrow is $x \rightarrow \delta_x$, point evaluations. Completing,

$$\begin{array}{ccc} & Y & \\ \nearrow & \tilde{e} \uparrow & \searrow^{J_Y} \\ \widehat{\Lambda} \mid & U & \rightarrow \widehat{\mathcal{F}(U)'_\alpha} \\ \searrow & e \downarrow & \nearrow_{\widehat{J}_X} \\ & \mathcal{F}_*(U) & \end{array}$$

$\widehat{\Lambda}$ is surjective: take $y \in Y$; recall that $[\tilde{e}(U)]$ is dense in Y , and let $\sigma_i = \sum_{x \in U} a_x^i \tilde{e}(x) \rightarrow y$. Thus, setting $s_i = \sum_{x \in U} a_x^i e_x$, we have $\Lambda(s_i) = \sigma_i$. Now $J_X(s_i) = J_Y(\Lambda(s_i)) = J_Y(\sigma_i) \rightarrow J_Y(y)$, and $(J_X(s_i))$ is a Cauchy net in $Im \widehat{J}_X$. But this is isomorphic to $\mathcal{F}_*(U)$ (indeed, $\widehat{J}_X : \mathcal{F}_*(U) \rightarrow Im \widehat{J}_X$ is a homeomorphism). Hence, (s_i) is a Cauchy net in $\mathcal{F}_*(U)$, so $s_i \rightarrow s$. Then

$$\sigma_i = \Lambda(s_i) \rightarrow \widehat{\Lambda}(s),$$

so $y = \widehat{\Lambda}(s)$. Hence $Im J_Y \subset Im \widehat{J}_X$, furthermore $\widehat{\Lambda}^{-1} = \widehat{J}_X^{-1} \circ J_Y$, and so is continuous. ■

Theorem 5.4. *Suppose Y is a complete locally convex space, $\tilde{e} : U \rightarrow Y$ continuous and each function in $\mathcal{F}(U)$ factors through \tilde{e} :*

$$\begin{array}{ccc} U & \rightarrow & \mathbb{C} \\ \tilde{e} \downarrow & \nearrow_T & \\ Y & & \end{array}$$

identifying Y' with $\mathcal{F}(U)$ algebraically. Suppose also that the following condition holds:

(*) *for each $y \in Y$ there is a bounded net $(x_i) \subset X$ such that $\Lambda(x_i)$ converges to y uniformly on $\sigma(\mathcal{F}(U), Y)$ -bounded subsets of $\mathcal{F}(U)$.*

Then $Y = \mathcal{F}_(U)$.*

Proof: We prove that $J_Y : Y \rightarrow \mathcal{F}(U)'_\alpha$ is continuous, and apply Lemma 5.3. Let $y_j \rightarrow y$ in Y . We want to see that $J_Y(y_j) \rightarrow J_Y(y)$ in $\mathcal{F}(U)'_\alpha$, i.e. that $T_f(y_j) \rightarrow T_f(y)$ uniformly (of f) on equicontinuous τ_p -compact disks in $\mathcal{F}(U)$. We consider X 'contained' in Y via Λ , and note that then equicontinuous τ_p -compact disks are equicontinuous $\sigma(\mathcal{F}(U), X)$ -compact disks, and by Lemma 5.2, equicontinuous $\sigma(\mathcal{F}(U), Y)$ -compact disks. Thus, we wish to prove that $T_f(y_j) \rightarrow T_f(y)$ uniformly (of f) on equicontinuous $\sigma(\mathcal{F}(U), Y)$ -compact disks.

Now, as in Proposition 5.1, uniform convergence on equicontinuous subsets of $\mathcal{F}(U)$ corresponds to the continuity of \tilde{e} , while uniform convergence over $\sigma(\mathcal{F}(U), Y)$ -compact disks (Mackey topology), corresponds to the fact that $Y' = \mathcal{F}(U)$. ■

Corollary 5.5. *If Y is a Banach space, $\tilde{e} : U \rightarrow Y$ continuous and each function in $\mathcal{F}(U)$ factors through \tilde{e} :*

$$\begin{array}{ccc} U & \rightarrow & \mathbb{C} \\ \tilde{e} \downarrow & \nearrow_T & \\ Y & & \end{array}$$

identifying Y' with $\mathcal{F}(U)$ algebraically, then $Y = \mathcal{F}_(U)$.*

Proof: The condition (*) in Theorem 5.4 is automatically verified if Y is a Banach space: $\sigma(\mathcal{F}(U), Y)$ -bounded sets are just the w^* -bounded sets, so uniform convergence on them is norm-convergence. But as we have seen in Theorem 3.3, $Im \Lambda$ is dense in Y , so for each $y \in Y$ there is a sequence $(x_n) \subset X$ such that $\Lambda(x_n) \rightarrow y$ in norm. The sequence (x_n) is weakly bounded, and thus bounded in X , because $X' = Y'$. ■

Corollary 4.3 and Corollary 5.5 provide us with two criteria which will enable us to identify previously constructed preduals with ours. We use these corollaries and Proposition 5.1 in the examples below.

Example 1. $P^k(E)$.

The space $P^k(E)$ of continuous k -homogeneous polynomials on a Banach space E . This space is a Banach space with the norm

$$\|P\| = \sup_{\{\|x\|\leq 1\}} |P(x)|.$$

R. Ryan [12] has proved that the symmetric projective tensor product $\widehat{\bigotimes}_{\pi,s,k} E$ is a predual of $P^k(E)$. Further, $\widehat{\bigotimes}_{\pi,s,k} E$ factors continuous k -homogeneous polynomials

$$\begin{array}{ccc} E & \rightarrow & \mathbb{C} \\ \tilde{e} \downarrow & \nearrow_L & \\ \widehat{\bigotimes}_{\pi,s,k} E & & \end{array}$$

where $\tilde{e}(x) = x \otimes \cdots \otimes x$. Since $\widehat{\bigotimes}_{\pi,s,k} E$ is a Banach space, by Corollary 5.5 we have that

$$\mathcal{F}_*(E) = \widehat{\bigotimes}_{\pi,s,k} E.$$

Also, we have $(\mathcal{F}_*(E)', \beta) = P^k(E)$, by Theorem 2.2. A subset of E is P^k -bounded if and only if it is norm-bounded. Thus the BBP^k -property mentioned above is exactly the $(BB)_k$ -property [4].

Example 2. $P_I^k(E)$.

The space $P_I^k(E)$ of continuous k -homogeneous integral polynomials on a Banach space E . These are the polynomials which can be written as

$$P(x) = \int_{B_{E'}} \gamma(x)^k d\mu(\gamma)$$

for some regular Borel measure on the unit ball $B_{E'}$ of E' . This space is a Banach space with the norm

$$\|P\|_I = \inf\{|\mu| : \text{such a representation holds}\}$$

S. Dineen [3] has proved that the symmetric injective tensor product $\widehat{\bigotimes}_{\epsilon,s,k} E$ is a predual of $P_I^k(E)$. Further, $\widehat{\bigotimes}_{\epsilon,s,k} E$ factors integral k -homogeneous polynomials

$$\begin{array}{ccc} E & \rightarrow & \mathbb{C} \\ \tilde{e} \downarrow & \nearrow_L & \\ \widehat{\bigotimes}_{\epsilon,s,k} E & & \end{array}$$

where $\tilde{e}(x) = x \otimes \cdots \otimes x$. As a consequence of Corollary 5.5 we have

$$\mathcal{F}_*(E) = \widehat{\bigotimes}_{\epsilon,s,k} E.$$

Note that $(\mathcal{F}_*(E)', \beta) = P_I^k(E)$. Indeed, $P_I^k(E)$ is barrelled, but when $\mathcal{F}_*(U)$ is a Banach space, the topologies β and β' coincide. A subset

of E is P_I^k -bounded if and only if it is norm-bounded ($\gamma^k \in P_I^k(E)$ for all $\gamma \in E'$). Thus the BBP_I^k -property mentioned above is analogous to the $(BB)_k$ -property [4]. In this case, E does not have this property.

Example 3. $H^\infty(U)$.

Let U be an open subset of a Banach space and $H^\infty(U)$ the space of bounded holomorphic functions on U . This is a Banach space when equipped with the supremum norm. J. Mujica [8] has constructed a Banach predual $G^\infty(U)$ of $H^\infty(U)$ factoring bounded holomorphic functions

$$\begin{array}{ccc} U & \rightarrow & \mathbb{C} \\ \delta_U \downarrow & \nearrow_L & \\ G^\infty(U) & & \end{array}$$

By Corollary 5.5,

$$\mathcal{F}_*(U) = G^\infty(U).$$

We have $(\mathcal{F}_*(U)', \beta) = H^\infty(U)$ by Theorem 2.2. Also, U has the BBH^∞ -property.

Example 4. $H_b(U)$.

Let U be an open balanced subset of a Banach space, and $H_b(U)$ the space of holomorphic functions of bounded type on U , that is, the functions which are bounded on subsets $V \subset U$ which are bounded and bounded away from the boundary of U . $H_b(U)$ is a Fréchet space with the seminorms

$$p_V(f) = \sup_V |f|.$$

In [5] P. Galindo, D. García, and M. Maestre have constructed an (LB) -space $\mathcal{P}_b(U)$ which is a predual of $H_b(U)$ and factors holomorphic functions of bounded type. Moreover, for a locally convex space F , the spaces $\omega H_b(U, F)$ and $H_b(U, F)$ coincide (note that a continuous weakly holomorphic function is holomorphic and a weakly bounded set is also bounded). Since $\mathcal{P}_b(U)$ factors functions in $H_b(U, F) = \omega H_b(U, F)$ (see [5] and [9]), by Corollary 4.3 we have that

$$\mathcal{F}_*(U) = \mathcal{P}_b(U).$$

$H_b(U)$ is a Fréchet (and therefore barrelled) space whose topology is that of uniform convergence on H_b -bounded sets (in this case, H_b -bounded is the same as bounded and bounded away from the boundary). By Theorem 2.2, $H_b(U) = (\mathcal{F}_*(U)', \beta)$. Also, U has the BBH_b -property.

Example 5. $H_I(B_E^\circ)$.

The space $H_I(B_E^\circ)$ of integral analytic functions on the open unit ball of a Banach space is the space of holomorphic functions which may be written

$$f(x) = \int_{B_{E'}} \frac{1}{1 - \gamma(x)} d\mu(\gamma)$$

for some regular Borel measure on the unit ball $B_{E'}$ of E' . This space is a Banach space with the norm

$$\|P\|_I = \inf\{|\mu| : \text{such a representation holds}\}.$$

It has been shown in [2] that when E has the approximation property, the ball algebra $A(B_{E'})$ is a predual of $H_I(B_E^\circ)$. This predual factors integral holomorphic functions through $g : B_E^\circ \rightarrow A(B_{E'})$, $g(x) = \frac{1}{1-x}$. Since the ball algebra is a Banach space, by Corollary 5.5 we have

$$\mathcal{F}_*(B_{E'}) = A(B_{E'}).$$

Note that here too $(\mathcal{F}_*(B_{E'})', \beta) = H_I(B_E^\circ)$.

Example 6. $H(U)$.

In [4], S. Dineen has constructed a predual $G(U)$ of $H(U)$ factoring holomorphic functions

$$\begin{array}{ccc} U & \rightarrow & \mathbb{C} \\ \delta_U \downarrow & \nearrow_L & \\ G(U) & & \end{array}$$

The topology on $G(U)$ is given by uniform convergence on locally bounded subsets of $H(U)$. As a consequence of Ascoli's Theorem and Montel's Theorem (see, for example, [4, Lemma 3.25]) and the characterization given in Proposition 5.1, this topology coincides with the topology α and therefore

$$\mathcal{F}_*(U) = G(U).$$

Since $(H(U), \tau_\delta)$ is a barrelled space [4, Prop. 3.18], $(H(U), \tau_\delta)$ and $(\mathcal{F}_*(U)', \beta')$ are isomorphic. If U is contained in a Banach space with a basis, $\tau_\delta = \tau_0$ in $H(U)$ [4, Cor. 4.16 and Prop. 5.7]. But $\tau_0 \leq \beta \leq \beta' = \delta$, so $(H(U), \tau_\delta) = (\mathcal{F}_*(U)', \beta)$.

Example 7. $C(U)$

Let U be a Hausdorff topological space. By Ascoli's Theorem, a subset of $(C(U), \tau_0)$ is relatively compact if and only if it is equicontinuous and pointwise bounded. Therefore, Proposition 5.1 affirms that the topology α is given by uniform convergence on τ_0 -compact subsets of $C(U)$. That is, the topology induced by the inclusion $\mathcal{F}_*(U) \hookrightarrow (C(U), \tau_0)'_{\tau_0}$.

Moreover, $(C(U), \tau_0)$ is barrelled if and only if C -bounded subsets of U are relatively compact [4, Ex. 3.77] and then τ_0 is the topology of uniform convergence on C -bounded subsets of U . By Theorem 2.2, in this case we have that $(\mathcal{F}_*(U)', \beta) = C(U)$, and also that U has the *BBC*-property.

Example 8. ℓ_2 .

Of the continuous functions $x : \mathbb{N} \rightarrow \mathbb{C}$, consider those for which $\sum_n |x_n|^2 < \infty$. We have ℓ_2 . The space X defined in section 1 is just the space of finite sequences $\mathbb{C}^{(\mathbb{N})}$, and α is the Mackey topology corresponding to the duality $\langle \mathbb{C}^{(\mathbb{N})}, \ell_2 \rangle$: uniform convergence on $\sigma(\ell_2, \mathbb{C}^{(\mathbb{N})})$ -compact disks $B \subset \ell_2$. By Lemma 5.2, this is the same topology as that of uniform convergence on $\sigma(\ell_2, \ell_2)$ -compact disks and therefore

$$\mathcal{F}_*(\mathbb{N}) = \ell_2.$$

Note that both compactness of B and the fact that it is a disk play an important role. Indeed, consider the $\sigma(\ell_2, \mathbb{C}^{(\mathbb{N})})$ -compact subset $B = \{ne_n : n \in \mathbb{N}\} \cup \{0\}$. Although $\frac{1}{k}e_k$ converges to 0 in $(\mathbb{C}^{(\mathbb{N})}, \|\cdot\|_2)$, we have that $p_B(\frac{1}{k}e_k) = 1$ for all $k \in \mathbb{N}$. So B is not taken into account for the Mackey topology.

Moreover, if we take $\text{coe}(B)$ the absolutely convex hull of B , $p_{\text{coe}(B)}$ is not a continuous seminorm for the Mackey topology, although $\text{coe}(B)$ is a $\sigma(\ell_2, \mathbb{C}^{(\mathbb{N})})$ -precompact disk in ℓ_2 . Observe that in order to $\sigma(\ell_2, \mathbb{C}^{(\mathbb{N})})$ -compactify the set $\text{coe}(B)$, we need to complete it for the $\sigma(\ell_2, \mathbb{C}^{(\mathbb{N})})$ topology. Unfortunately, that completion will not remain inside ℓ_2 . To see this, consider $a_n = \sum_{k=1}^n \frac{1}{2^k} (2^k e_{2^k}) \in \text{coe}(B)$. If a subnet of $(a_n)_n$ converges to some a in the $\sigma(\ell_2, \mathbb{C}^{(\mathbb{N})})$ topology, we have that $\langle e_{2^n}, a \rangle = 1$ for all $n \in \mathbb{N}$ and a cannot belong to ℓ_2 . Therefore $\text{coe}(B)$ is not one of Mackey's selected subsets.

Example 9. ℓ_1 .

Consider now the space ℓ_1 . X is $\mathbb{C}^{(\mathbb{N})}$, and α is the Mackey topology corresponding to the duality $\langle \mathbb{C}^{(\mathbb{N})}, \ell_1 \rangle$. Again by Lemma 5.2, this topology is given by the supremum norm, and the predual we have constructed is

$$\mathcal{F}_*(\mathbb{N}) = c_0.$$

Note that by Corollary 5.5, any other Banach space 'linearizing' ℓ_1 must be isomorphic to c_0 . A case in point is the space c which corresponds to the linearizing map $\tilde{e} : \mathbb{N} \rightarrow c$ given by

$$\tilde{e}(n) = \begin{cases} (1, 1, \dots), & \text{if } n = 1. \\ e_{n-1}, & \text{if } n > 1. \end{cases}$$

Since there exist Banach preduals of ℓ_1 which are not isomorphic to c_0 , such preduals will not constitute linearizations in the sense we have defined.

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