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"Asymptotic Behavior of the Eigenvalues of the One Dimensional Weighted p-Laplace Operator"

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ASYMPTOTIC BEHAVIOR OF THE EIGENVALUES OF THE ONE DIMENSIONAL WEIGHTED $p-$LAPLACE OPERATOR

JULIÁN FERNÁNDEZ BONDER AND JUAN PABLO PINASCO

Abstract. In this paper we study the spectral counting function for the weighted p-laplacian in one dimension. We show the existence of a Weyl-type leading term and we find estimates for the remainder term.

1. Introduction

In this paper we study the following eigenvalue problem:

$$-(\psi_p(u'))' = \lambda r(x)\psi_p(u), \tag{1.1}$$

in a bounded open set $\Omega \subset \mathbb{R}$, with Dirichlet or Neumann boundary conditions. Here, the weight r is a real-valued, bounded, positive continuous function, λ is a real parameter and

$$\psi_p(s) = |s|^{p-2}s,$$

for $s \neq 0$ and 0 if $s = 0$.

From [14] we know that the spectrum consists on a countable sequence of nonnegative eigenvalues $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ (repeated according multiplicity) tending to $+\infty$. With the same ideas as in [2], Theorem 4.1 it is easy to prove that the sequence $\{\lambda_k\}_k$ coincide with the eigenvalues obtained by the Ljusternik-Schnirelmann theory. We define the spectral counting function $N(\lambda, \Omega)$ as the number of eigenvalues of problem (1.1) less than a given λ :

$$N(\lambda, \Omega) = \#\{k \,:\, \lambda_k \leq \lambda\}.$$

We will write $N_D(\lambda, \Omega)$ (resp., $N_N(\lambda, \Omega)$) whenever we need to stress the dependence on the Dirichlet (resp., Neumann) boundary conditions.

We obtain the following asymptotic expansion:

$$N(\lambda, \Omega) \sim \frac{\lambda^{1/p}}{\pi_p} \int_\Omega r^{1/p}, \tag{1.2}$$

as $\lambda \to \infty$, where π_p is defined as

$$\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}. \tag{1.3}$$

The proof is based on variational arguments, including a suitable extension of the method of 'Dirichlet-Neumann bracketing' in [1].

Moreover, we analyze the remainder term $R(\lambda, \Omega) = N(\lambda, \Omega) - \frac{1}{\pi_p} \int_\Omega (\lambda r)^{1/p}$, following the ideas on [4]. We show that

$$R(\lambda, \Omega) = O(\lambda^{\delta/p}), \tag{1.4}$$

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where $\delta \in (0,1]$ depends on the regularity of the boundary $\partial\Omega$.

When $p = 2$, the problem has a long history, see [4, 7, 8, 10] and the references therein.

For $p \neq 2$, the only known result is due to Garcia Azorero and Peral Alonso, [6]. The authors show that the eigenvalues of the p-laplacian in $\mathbb{R}^N$ (with $r = 1$) obtained by the mini-max theory satisfy

$$c_1(\Omega)k^{p/N} \le \lambda_k \le c_2(\Omega)k^{p/N}. \tag{1.5}$$

It is easy to see that this eigenvalue inequality is equivalent to

$$C_1(\Omega)\lambda^{N/p} \le N(\lambda,\Omega) \le C_2(\Omega)\lambda^{N/p},$$

for certain positive constants when $\lambda \to \infty$, see Lemma 3.2 below.

Finally, the asymptotic behavior of the eigenvalues in [6] may be improved using the Dirichlet-Neumann bracketing. For $n = 1$, we obtain

$$\lambda_k \sim ck^p.$$

The paper is organized as follows. In §2, we introduce the genus in a version due to Krasnoselski and we prove some auxiliary lemmas. In §3, we prove the asymptotic expansion (1.2). We analyze the remainder estimate in §4. Finally, in §5, we compute explicitly a non-trivial second term for $r = 1$ and analyze the asymptotic behavior of the eigenvalues.

2. Preliminary results

In this section we introduce the main tools to deal with our problem, the genus and the Dirichlet-Neumann bracketing.

We want to remark that the results of this section holds for arbitrary dimensions $N \ge 1$ if one consider only the variational eigenvalues.

Let X be a Banach space. We consider the class:

$$\Sigma = \{A \subset X \ : \ A \ \text{compact} \ , \ A = -A\}.$$

We recall the genus $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$ as

$$\gamma(A) = \min\{k \in \mathbb{N} \ \text{ there exist } \ f \in C(A, \mathbb{R}^k \setminus \{0\}), \ f(x) = -f(-x)\}.$$

For some properties of the genus and some of its applications we refer to [15].

By the Ljusternik-Schnirelmann theory, there exists a sequence of nonlinear eigenvalues of problem (1.1) with Dirichlet (resp. Neumann) boundary condition, given by

$$\lambda_k^\Omega = \inf_{F \in C_k^\Omega} \sup_{u \in F} \int_\Omega |u'|^p, \tag{2.1}$$

where

$$C_k^\Omega = \left\{C \subset M^\Omega \ : \ C \ \text{compact} \ , \ C = -C, \ \gamma(C) \ge k\right\},$$
$$M^\Omega = \left\{u \in W_0^{1,p}(\Omega) \ (\text{ resp., } W^{1,p}(\Omega) \ ) \ : \ \int_\Omega r(x)|u|^p = 1\right\}.$$

Theorem 2.1. *Let $U_1, U_2 \in \mathbb{R}^N$ be disjoint open sets such that $(\overline{U_1 \cup U_2})^{int} = U$ and $|U \setminus U_1 \cup U_2|_N = 0$, then*

$$N_D(\lambda, U_1 \cup U_2) \le N_D(\lambda, U) \le N_N(\lambda, U) \le N_N(\lambda, U_1 \cup U_2).$$

Here $|A|_N$ stands for the N-dimensional Lebesgue measure of the set A.

Proof. It is an easy consequence of the following inclusions

$$W_0^{1,p}(U_1 \cup U_2) = W_0^{1,p}(U_1) \oplus W_0^{1,p}(U_2) \subset W_0^{1,p}(U) \tag{2.2}$$

and

$$W^{1,p}(U) \subset W^{1,p}(U_1) \oplus W^{1,p}(U_2) = W^{1,p}(U_1 \cup U_2), \tag{2.3}$$

and the variational formulation (2.1). In fact, using that

$$M^U(X) = \left\{ u \in X \ : \ \int_U r(x)|u|^p = 1 \right\} \subset M^U(Y) = \left\{ u \in Y \ : \ \int_U r(x)|u|^p = 1 \right\},$$

and that $C_k^U(X) \subset C_k^U(Y)$ where $X = W_0^{1,p}(U_1 \cup U_2)$ or $W^{1,p}(U)$ and $Y = W_0^{1,p}(U)$ or $W^{1,p}(U_1 \cup U_2)$ respectively, we obtain the desired inequality. □

The Dirichlet-Neumann bracketing is a powerful tool combined with the following result:

Proposition 2.2. *Let $\Omega = \cup_j \Omega_j$, where $\{\Omega_j\}_j$ is a pairwise disjoint family of bounded open sets in $\mathbb{R}^N$. Then,*

$$N(\lambda, \Omega) = \sum_j N(\lambda, \Omega_j). \tag{2.4}$$

Proof. Let λ be an eigenvalue of problem (1.1) in Ω, and let u be the associated eigenfunction. For all $v \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v - \lambda \int_\Omega |u|^{p-2}uv = 0. \tag{2.5}$$

Choosing v with compact support in Ω_j, we conclude that $u|_{\Omega_j}$ is an eigenfunction of problem (1.1) in Ω_j with eigenvalue λ.

For the other inclusion, it is sufficient to extend an eigenfunction u in Ω_j by zero outside, which gives an eigenfunction in Ω. □

3. The function $N(\lambda)$

In this section we prove the asymptotic expansion (1.2).

Lemma 3.1. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the eigenvalues of (1.1) in $(0,T)$, with $r = 1$. Then,*

$$\lambda_k = \frac{\pi_p^p}{T^p} k^p. \tag{3.1}$$

Proof. This result was proved in [12]. □

Lemma 3.2. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the eigenvalues of (1.1) in $(0,T)$ and suppose that $m \le r(x) \le M$. Then,*

$$\frac{1}{M}\frac{\pi_p^p}{T^p} k^p \le \lambda_k \le \frac{1}{m}\frac{\pi_p^p}{T^p} k^p, \tag{3.2}$$

and

$$\frac{Tm^{1/p}}{\pi_p}\lambda^{1/p} - 1 \le N(\lambda, (0,T)) \le \frac{TM^{1/p}}{\pi_p}\lambda^{1/p}. \tag{3.3}$$

Proof. Equation (3.2) is an easy consequence of the Sturmian Comparison principle in [13] or [14] and the explicit formula for the eigenvalues. Now,

$$\#\left\{k : \frac{\pi_p^p k^p}{T^p M} \le \lambda\right\} \le \#\{k : \lambda_k \le \lambda\} \le \#\left\{k : \frac{\pi_p^p k^p}{T^p m} \le \lambda\right\}. \tag{3.4}$$

The left hand side is greater than

$$\frac{Tm^{1/p}}{\pi_p}\lambda^{1/p} - 1,$$

which gives the lower bound. In the same way, we obtain

$$N(\lambda, (0,T)) \le \left[\frac{Tm^{1/p}}{\pi_p}\lambda^{1/p}\right] \le \frac{Tm^{1/p}}{\pi_p}\lambda^{1/p}.$$

The proof is complete. □

Remark 3.3. Sometimes is better to bound $x - [x]$ as x instead of $O(1)$, in order to obtain a convergence result, as in Theorem 3.5 below, or in the second example of Section 5.

Proposition 3.4. *Let $r(x)$ be a real-valued, positive continuous function in $[0,T]$. Then,*

$$N(\lambda, (0,T)) = \frac{\lambda^{1/p}}{\pi_p}\int_0^T r^{1/p} + o(\lambda^{1/p}). \tag{3.5}$$

Proof. Let $[0,T] = \overline{\cup_{1\le j\le J} I_j}$, $I_j \cap I_k = \emptyset$ with $|I_j| = T/J = \eta$. We define

$$m_j = \inf_{x\in I_j} r(x), \qquad M_j = \sup_{x\in I_j} r(x).$$

We can choose $\eta > 0$ such that

$$\sum_{j=1}^J \eta m_j^{1/p} = \int_0^T r^{1/p} - \varepsilon_1, \qquad \sum_{j=1}^J \eta M_j^{1/p} = \int_0^T r^{1/p} + \varepsilon_2,$$

with $\varepsilon_1, \varepsilon_2 > 0$ arbitrarily small.

From Theorem 2.1 and Proposition 2.2, we obtain

$$\sum_{j=1}^J N_D(\lambda, I_j) \le N(\lambda, (0,T)) \le \sum_{j=1}^J N_N(\lambda, I_j).$$

Hence, using that

$$N_D(\lambda, I_j) \ge m_j^{1/p}\frac{\lambda^{1/p}}{\pi_p} - 1 \quad \text{and} \quad N_N(\lambda, I_j) \le M_j^{1/p}\frac{\lambda^{1/p}}{\pi_p},$$

we have

$$\frac{\lambda^{1/p}}{\pi_p}\left(\int_0^T r^{1/p} - \varepsilon_1\right) - J \le N(\lambda, (0,T)) \le \frac{\lambda^{1/p}}{\pi_p}\left(\int_0^T r^{1/p} + \varepsilon_2\right).$$

Letting $\lambda \to \infty$, we have

$$\frac{N(\lambda, (0,T))}{\frac{\lambda^{1/p}}{\pi_p}\int_0^T r^{1/p}} \to 1$$

and the proof is complete. □

Now we prove the main Theorem of this section:

Theorem 3.5. *Let $r(x)$ be a real-valued, positive and bounded continuous function in Ω. Then,*

$$N(\lambda,\Omega) = \frac{\lambda^{1/p}}{\pi_p}\int_\Omega r^{1/p} + o(\lambda^{1/p}). \tag{3.6}$$

Proof. It is an easy consequence of Proposition 2.2 and Proposition 3.4. Let $\Omega = \cup_{j=1}^{\infty} I_j$, then

$$N(\lambda,\Omega) = \sum_{j=1}^{\infty} N(\lambda, I_j) \sim \sum_{j=1}^{\infty} \frac{\lambda^{1/p}}{\pi_p}\int_{I_j} r^{1/p} = \frac{\lambda^{1/p}}{\pi_p}\int_\Omega r^{1/p}. \tag{3.7}$$

This completes the proof. □

4. Remainder estimates

In order to get better asymptotic results, we will put some restrictions on Ω and r. Given any $\eta > 0$ sufficiently small, we consider a tessellation of $\mathbb{R}$ by a countable family of disjoint open intervals $\{I_\zeta\}_{\zeta\in\mathbb{Z}}$, of length η.

Definition 4.1. *Let Ω be a bounded open set in $\mathbb{R}$. Given $\beta > 0$, we say that the boundary $\partial\Omega$ satisfies the "β-condition" if there exist positive constants c_0 and $\eta_0 < 1$ such that for all $\eta \le \eta_0$,*

$$\frac{\#(J \setminus I)}{\# I} \le c_0 \eta^\beta, \tag{4.1}$$

where

$$I = I(\Omega) = \{\zeta \in \mathbb{Z} \ : \ I_\zeta \subset \Omega\}, \tag{4.2}$$

$$J = J(\Omega) = \{\zeta \in \mathbb{Z} \ : \ I_\zeta \cap \overline{\Omega} \neq \emptyset\}. \tag{4.3}$$

It is easy to see that if the set is Jordan contented (i.e., it is well approximated from within and without by a finite union of intervals), then it verifies the "β-condition" for $\beta = 1$. The coefficient β allows us to measure the smoothness of $\partial\Omega$.

Definition 4.2. *Given $\gamma > 0$, we say that the function r satisfies the "γ-condition" if there exist positive constants c_1 and $\eta_1 < 1$ such that for all $\zeta \in I(\Omega)$ and all $\eta \le \eta_1$,*

$$\int_{I_\zeta} |r - r_\zeta|^{1/p} \le c_1 \eta^\gamma, \tag{4.4}$$

where $r_\zeta = \left(|I_\zeta|^{-1}\int_{I_\zeta} r^{1/p}\right)^p$ is the mean value of $r^{1/p}$ in I_ζ.

Remark 4.3. 1. The coefficient γ enable us to measure the smoothness of r, the larger γ, the smoother r.

2. When r is Holder continuous of order $\theta > 0$ and is bounded away from zero on Ω, then it satisfies the γ-condition for $0 < \gamma \le 1 + \theta/p$.

If r is only continuous and positive on $\overline{\Omega}$, then it satisfies the γ-condition for $0 < \gamma \le 1$

We can now state the main theorem of this section:

Theorem 4.4. *Let Ω be a bounded open set in $\mathbb{R}$ with boundary $\partial\Omega$ satisfying the "β-condition" for some $\beta > 0$, and let r be a bounded, positive and continuous function satisfying the "γ-condition" for some $\gamma > 1$. Set $\nu = \min(\beta, \gamma - 1)$. Then, for all $\delta \in [1/(\nu+1), 1]$, we have*

$$N(\lambda, \Omega) - \frac{1}{\pi_p} \int_\Omega (\lambda r)^{1/p} = O(\lambda^{\delta/p}) \tag{4.5}$$

Proof. For the convenience of the reader, the proof is divided into several steps.

Moreover, we will stress the dependence of problem (1.1) in the weight function, writing $N(\lambda, \Omega, f)$.

Step 1. Let $\eta > 0$ be fixed. We define

$$\varphi(\lambda) = \pi_p^{-1} \int_\Omega (\lambda r)^{1/p}, \qquad \varphi(\lambda, \zeta) = \eta \pi_p^{-1} (\lambda r_\zeta)^{1/p}. \tag{4.6}$$

From Theorem 2.1 we obtain

$$\sum_{\zeta \in I} N_D(\lambda, I_\zeta, r) - \varphi(\lambda) \le N_D(\lambda, \Omega, r) - \varphi(\lambda) \tag{4.7}$$

and

$$N_D(\lambda, \Omega, r) - \varphi(\lambda) \le \sum_{\zeta \in I} N_N(\lambda, I_\zeta, r) + \sum_{\zeta \in J \setminus I} N_N(\lambda, I_\zeta \cap \Omega, r) - \varphi(\lambda). \tag{4.8}$$

We are reduced to find a bound for the left (resp., right) term of (4.7) (resp., (4.8)).

Step 2. We can rewrite (4.7) as:

$$\begin{aligned} \sum_{\zeta \in I} N_D(\lambda, I_\zeta, r) - \varphi(\lambda) \le & \sum_{\zeta \in I} N_D(\lambda, I_\zeta, r_\zeta) - \varphi(\lambda, \zeta) + \sum_{\zeta \in I} \varphi(\lambda, \zeta) - \varphi(\lambda) \\ & + \sum_{\zeta \in I} N_D(\lambda, I_\zeta, r) - \sum_{\zeta \in I} N_D(\lambda, I_\zeta, r_\zeta). \end{aligned} \tag{4.9}$$

Let us note that both $\sum_{\zeta \in I} N_D(\lambda, I_\zeta, r_\zeta) - \varphi(\lambda, \zeta)$ and $\sum_{\zeta \in I} \varphi(\lambda, \zeta) - \varphi(\lambda)$ are negative. Now, by Lemma 3.2:

$$\sum_{\zeta \in I} |N_D(\lambda, I_\zeta, r_\zeta) - \varphi(\lambda, \zeta)| \le \#(I) M \le \eta^{-1} |\Omega|. \tag{4.10}$$

We can bound

$$\sum_{\zeta \in I} \varphi(\lambda, \zeta) - \varphi(\lambda) = \pi_p^{-1} \lambda^{1/p} \left(\sum_{\zeta \in I} \int_{I_\zeta} (r^{1/p} - r_\zeta^{1/p}) + \sum_{\zeta \in J \setminus I} \int_{I_\zeta \cap \Omega} r^{1/p} \right)$$

as

$$C \lambda^{1/p} \#(J \setminus I) \eta M \le C \lambda^{1/p} \eta^\beta. \tag{4.11}$$

Here we have used that $r \le M$, and that $\partial\Omega$ satisfies the β-condition.

Finally, the third term in (4.9) can be handled using the monotonicity of the eigenvalues with respect to the weight (see [14]). Replacing $r \le r_\zeta + |r - r_\zeta|$, a simple computation shows that

$$N(\lambda, I_\zeta, r) \le N(\lambda, I_\zeta, r_\zeta) + N(\lambda, I_\zeta, |r - r_\zeta|),$$

which gives

$$\sum_{\zeta \in I} N_D(\lambda, I_\zeta, r) - N_D(\lambda, I_\zeta, r_\zeta) \le \sum_{\zeta \in I} N(\lambda, I_\zeta, |r - r_\zeta|) \le C \lambda^{1/p} \#(I) \eta^\gamma$$

and using the same arguments as above and the fact that r satisfies the γ-condition, we obtain

$$\sum_{\zeta\in I} N_D(\lambda, I_\zeta, r) - N_D(\lambda, I_\zeta, r_\zeta) \leq C\lambda^{1/p}\eta^{\gamma-1}. \tag{4.12}$$

Collecting (4.10), (4.11) and (4.12) we have the lower bound

$$C\lambda^{1/p}(\eta^\beta + \eta^{\gamma-1}) + C\eta^{-1}. \tag{4.13}$$

Step 3. In a similar way, we can find an upper bound for (4.8),

$$\left(\sum_{\zeta\in I} N_N(\lambda, I_\zeta, r) - \varphi(\lambda)\right) + \sum_{\zeta\in J\setminus I} N_N(\lambda, I_\zeta\cap\Omega, r). \tag{4.14}$$

We only need to estimate the last term, but

$$N_N(\lambda, I_\zeta\cap\Omega, r) \leq C\lambda^{1/p}\int_{I_\zeta\cap\Omega} r^{1/p} \leq C(M\eta\lambda)^{1/p}$$

and again, using the β-condition, we have

$$\sum_{\zeta\in J\setminus I} N_N(\lambda, I_\zeta\cap\Omega, r) \leq C\lambda^{1/p}\eta^\beta. \tag{4.15}$$

Hence, we obtain an upper bound for (4.8):

$$C\lambda^{1/p}(\eta^\beta + \eta^{\gamma-1}) + C\eta^{-1}. \tag{4.16}$$

Step 4. From (4.13) and (4.16) we obtain

$$|N(\lambda,\Omega) - \frac{1}{\pi_p}\int_\Omega (\lambda r)^{1/p}| \leq C\lambda^{1/p}(\eta^\beta + \eta^{\gamma-1}) + C\eta^{-1}. \tag{4.17}$$

We now choose $\eta = \lambda^{-a}$, with $0 < a \leq \delta$. It is clear that the last term in (4.17) is bounded by $C\lambda^\delta$. Also, it is easy to see that, if $a \geq \frac{1}{\beta}(\frac{1}{p} - \delta)$, then $\lambda^{1/p}\eta^\beta \leq \lambda^\delta$. Likewise, choosing $a \geq \frac{1}{\gamma-1}(\frac{1}{p} - \delta)$, we have $\lambda^{1/p}\eta^{\gamma-1} \leq \lambda^\delta$. When $\beta = 0$, or $\gamma = 1$, we must choose $a = 1/p$.

This completes the proof. □

5. Concluding remarks

We end this paper showing a family of examples with a power-like second term, and an example with an irregular second term. Finally, we discuss the asymptotic behavior of the eigenvalues.

In the examples below, the parameter d provides some geometrical information about $\partial\Omega$. In both cases, d is the interior Minkowski (or box) dimension of the boundary, we refer the reader to [3] and references therein.

Examples of explicit second term. Let $\Omega = \cup_j I_j$, where $|I_j| = j^{-1/d}$, and $0 < d < 1$. We have the following asymptotic expansion for the spectral counting function when $r = 1$:

$$N(\lambda, \Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + C(d)\lambda^{d/p} + O(\lambda^{d/p(2+d)}). \tag{5.1}$$

The proof can be obtained with number-theoretic methods. We have:

$$N(\lambda, \Omega) = \sum_{j=1}^{\infty} \left[\frac{j^{-1/d}}{\pi_p}\lambda^{1/p}\right] = \#\{(m,n) \in \mathbb{N}^2 \ : \ m.n^{1/d} \le \pi_p^{-1}\lambda^{1/p}\}.$$

In fact, for each j we can draw the vertical segment of length $j^{-1/d}\lambda^{1/p}/\pi_p$ in the plane, and the series in the left is the number of lattice points below the function $y(x) = \frac{\lambda^{1/p}}{\pi_p}x^{-1/d}$. See [11] for a detailed proof.

When $p = 2$ and $|I_j| \sim j^{-1/d}$, Lapidus and Pomerance in [9] showed that

$$N(\lambda, \Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} + C(d)\lambda^{d/p} + o(\lambda^{d/p}),$$

without the lattice point theory, the same result is valid for $p \ne 2$. However, let us note that the error in equation (5.1) is better, which enables us to obtain more precise estimates whenever we know more about the asymptotic behavior of $|I_j|$. On the other hand, the result in [9] holds for more general domains that the ones considered here.

Example of irregular second term. Let Ω be the complement of the ternary Cantor set, and $r = 1$. We have:

$$N(\lambda, \Omega) = \frac{|\Omega|}{\pi_p}\lambda^{1/p} - f(\ln(\lambda))\lambda^{\ln(2)/p\ln(3)} + O(1). \tag{5.2}$$

Here $f(x)$ is a bounded, periodic function. Our proof follows closely [5], where the usual Laplace operator on a self-similar set in $\mathbb{R}^n$ was studied for every $n \ge 2$.

Let us define $\rho(x) = x - [x]$, it is evident that $|\rho(x)| \le \min(x, 1)$. Hence,

$$N(\lambda, \Omega) - \frac{|\Omega|}{\pi_p}\lambda^{1/p} = -\sum_{j=0}^{\infty} 2^j \rho\left(\frac{\lambda^{1/p}}{3^{j+1}\pi_p}\right) \le C\lambda^{1/p}. \tag{5.3}$$

It remains to prove the periodicity of f. We write the error term as

$$\sum_{j=-\infty}^{\infty} 2^j \rho\left(\frac{\lambda^{1/p}}{3^{j+1}\pi_p}\right) - \sum_{j=-\infty}^{-1} 2^j \rho\left(\frac{\lambda^{1/p}}{3^{j+1}\pi_p}\right). \tag{5.4}$$

Using that $|\rho(x)| \le 1$, the second series converges and it is bounded by a constant.

Let us introduce the new variable

$$y = \frac{\ln(\lambda^{1/p}) - \ln(\pi_p)}{\ln(3)}, \tag{5.5}$$

which gives $3^y = \lambda^{1/p}/\pi_p$ and $2^y = (\lambda^{1/p}/\pi_p)^d$, where

$$d = \frac{\ln(2)}{\ln(3)}. \tag{5.6}$$

Replacing in (5.4), we obtain

$$\frac{1}{2}\sum_{j=-\infty}^{\infty} 2^j \rho\left(\frac{\lambda^{1/p}}{3^j \pi_p}\right) = \frac{1}{2}\left(\frac{\lambda^{1/p}}{3^j \pi_p}\right)^d \sum_{j=-\infty}^{\infty} 2^{j-y}\rho(3^{y-j}). \tag{5.7}$$

Thus, as $j-(y-1)=(j+1)-y$, we deduce that $f(x)$ is periodic with period equal to one.

Asymptotics of eigenvalues. From Theorem (3.6) it is easy to prove the following asymptotic formula for the eigenvalues:

$$\lambda_k \sim ck^p.$$

It follows immediately since $k \sim N(\lambda_k)$, which gives:

$$\lambda_k \sim \left(\frac{\pi_p}{\int_\Omega r^{1/p}}\right)^p k^p.$$

Using the Dirichlet-Neumann bracketing, it is possible to improve the constants in equation (1.5). In [6] the authors only consider two cubes $Q_1 \subset \Omega \subset Q_2$, and they obtain a lower and an upper bound for the eigenvalues in cubes which depends on the measure of the cubes Q_1, Q_2 instead of the measure of Ω.

A similar argument as in [6], changing the functions $\{\sin(kx)\}_k$ for $\{\sin_p(kx)\}_k$, gives the upper bound:

$$\lambda^k \leq \left(\frac{\pi_p}{|\Omega|}\right)^{p/n} k^{p/n}.$$

However, it seems difficult to improve the lower bound obtained with the aid of the Bernstein's Lemma.

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