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SPECTRAL ASYMPTOTICS OF THE NEUMANN LAPLACIAN ON OPEN SETS WITH FRACTAL BOUNDARIES

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ABSTRACT. We study the remainder estimate for the asymptotics of the number of eigenvalues of the Neumann laplacian in a bounded open set $\Omega \in \mathbb{R}^n$ with fractal boundary. We improve the previous results, showing that the Minkowski dimension and content should be replaced by the interior Minkowski dimension and content.

1. INTRODUCTION

Let Ω be an open bounded set in $\mathbb{R}^n$ with finitely many connected components. We consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial \eta}$ denotes a "normal derivative" along $\partial\Omega$. We interpret problem (1.1) in the variational sense, i.e., we say that λ is an eigenvalue if there exists a nonzero $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \nabla v = \lambda \int_\Omega uv \qquad \forall v \in H^1(\Omega) \tag{1.2}$$

We assume through the paper that the spectrum is discrete. We will discuss below some hypothesis on Ω to ensure that the spectrum is a countable set without accumulation points. Let $\{\lambda_j\}_j$ be the eigenvalues of the Neumann laplacian where $0 = \lambda_1 < \lambda_2 \leq \dots$ (repeated according to their multiplicity). Let $N(\lambda) = \#\{j : \lambda_j < \lambda\}$ be the associated spectral counting function. We are interested in the asymptotic behaviour of $N(\lambda)$.

When $\partial\Omega$ satisfies the so-called C-condition (see below), Metivier [Met1] showed that

$$N(\lambda) = (1 + o(1))\varphi(\lambda) \tag{1.3}$$

as $\lambda \to \infty$. Here, $\varphi(\lambda) = (2\pi)^{-n}\omega_n|\Omega|_n\lambda^{n/2}$, $|A|_n$ denotes the n-dimensional Lebesgue measure (or volume) of $A \subset \mathbb{R}^n$, and ω_n is the volume of the unit ball in R^n. For smooth boundaries and under some other geometric assumptions ([Iv, Ph]), one has a second term when $\lambda \to \infty$:

$$N(\lambda) = \varphi(\lambda) + c_n|\partial\Omega|_{n-1}\lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}), \tag{1.4}$$

where c_n is a constant which depends only on n.

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For the Dirichlet laplacian, in 1979 M. Berry [Br] made the following conjecture for $\partial\Omega$ with Hausdorff fractal dimension h:

$$N(\lambda) = \varphi(\lambda) - c_{h,n}\mu_h(\partial\Omega)\lambda^{h/2} + o(\lambda^{h/2}), \tag{1.5}$$

The conjecture cannot be true with h being the Hausdorff dimension, [B-C], and the authors suggested to replace h by D_M, the Minkowski (or box) dimension of the boundary.

In [Lap], the following asymptotic development was proved for the Dirichlet laplacian:

$$N(\lambda) = \varphi(\lambda) + O(\lambda^{d/2}) \tag{1.6}$$

as $\lambda \to \infty$, with d the interior Minkowski dimension.

For the Neumann laplacian, Lapidus obtained a similar result replacing d by D_M. This result was proved with a suitable extension of the Dirichlet-Neumann bracketing techniques, combined with precise estimates on the growth of the number of cubes in the tessellations of Ω. Moreover, he conjectured:

Conjecture 1.1. *Let Ω be a bounded open set of R^n with boundary $\partial\Omega$ satisfying either the C-condition or the "extension property". Assume that $\partial\Omega$ is Minkowski measurable and that D_M belongs to the open interval $(n-1, n)$. Then, for the Neumann laplacian, we have:*

$$N(\lambda) = \varphi(\lambda) + c_{D_M,n}M_{D_M}(\partial\Omega)\lambda^{D_M/2} + o(\lambda^{D_M/2}), \tag{1.7}$$

where $c_{D_M,n}$ is a positive constant depending only on n and D_M, and $M_{D_M}(\partial\Omega)$ denotes the Minkowski measure of $\partial\Omega$.

In this paper we show that D_M should be replaced by d, the interior Minkowski dimension. Namely, we prove that

$$N(\lambda) = \varphi(\lambda) + O(\lambda^{d/2}) \tag{1.8}$$

as $\lambda \to \infty$, as in the Dirichlet problem. We believe that the method used in [Lap] suggest the Minkowski dimension as the parameter involved in the spectral asymtotic. Instead of the Dirichlet -Neumann bracketing, we use the wave equations techniques. Our proof follows the ideas of Hormander [Ho1] and Guillemin, [Gui].

The paper is organized as follows: In section 2, we introduce the necessary notation and definitions, and we give a precise formulation of our results. Finally, in section 3 we prove the main theorem.

2. Hypothesis and main results.

Let A_ε denote the tubular neighborhood of radius ε of a set $A \subset \mathbb{R}^n$, i. e.,

$$A_\varepsilon = \{x \in \mathbb{R} : dist(x, A) \leq \varepsilon\} \tag{2.1}$$

We define the Minkowski dimension D_M of $\partial\Omega$ as

$$D_M = Dim_M(\partial\Omega) = \inf\{\delta \geq 0 \ : \ \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-\delta)}|(\partial\Omega)_\varepsilon|_n = 0\} \tag{2.2}$$

In a similar way, we define the interior Minkowski dimension as

$$d = dim(\partial\Omega) = \inf\{\delta \geq 0 \ : \ \limsup_{\varepsilon\to 0^+} \varepsilon^{-(n-\delta)}|(\partial\Omega)_\varepsilon \cap \Omega|_n = 0\} \tag{2.3}$$

It is evident that $d \leq D_M$. Moreover, even when $|\partial\Omega|_n$ is positive, the interior Minkowski dimension can be strictly lower than n.

We define the interior Minkowski content of $\partial\Omega$ as the limit (whenever it exist):

$$M_{int}(\partial\Omega, d) = \lim_{\varepsilon\to 0^+} \varepsilon^{-(n-d)} |(\partial\Omega)_\varepsilon \cap \Omega|_n. \tag{2.4}$$

Respectively, $M^*_{int}(\partial\Omega, d)(M_{*int}(\partial\Omega, d)$ denotes the $d-$dimensional upper (lower) interior Minkowski content, replacing the limit in (2.4) by an upper (resp., lower) limit.

It is not known a geometrical condition on Ω equivalent to the discreteness of the spectrum, however, we can impose the following condition in Ω (see [Met1]) in order to ensure it:

Definition 2.1. The open set Ω satisfies the C-condition if there exist positive constants ε_0, M, t_0 with $\varepsilon_0 M < t_0$, an open cover $\{\Omega_j\}_{1\leq j\leq N}$ of $\partial\Omega_\varepsilon$, and nonzero vectors h_j $(1 \leq j \leq N)$ in $\mathbb{R}^n$ such that $\forall j$, $\forall (x, y) \in \Omega_j \times \Omega_j$ with $|x - y| < \varepsilon_0$, and $\forall t \in \mathbb{R}$ with $M|x - y| \leq t \leq t_0$, the line segments $[x, x + th_j], [y, y + th_j]$ and $[x + th_j, y + th_j]$ are all contained in Ω.

Condition C holds in many cases, for instance if Ω satisfies the so-called "segment condition" or the "cone condition", or if $\partial\Omega$ is locally Lipschitz. See [F-M] or [Met2] for details.

Another condition is the "extension property" (i.e., the existence of a continuous linear extension map from $H^m(\Omega)$ to $H^m(\mathbb{R}^n)$). Under such hypothesis, it is possible to ensure that the embedding mapping from $H^1(\Omega)$ to $L^2(\Omega)$ is compact and as a consequence that the spectrum is discrete. Sufficient condition for the domain Ω to satisfy the "extension property" are obtained by Jones in [Jn].

Let us state our main results:

Theorem 2.2. *Let $\Omega \in \mathbb{R}^n$ be an open, bounded set, and $d \in [n-1, n]$ such that $M^*_{Int}(\partial\Omega, d) < +\infty$. We have the following estimates*
i) If $d \in (n-1, n]$, then:

$$N(\lambda, \Omega) = \varphi(\lambda) + O(\lambda^{d/2}) \tag{2.5}$$

when $\lambda \to \infty$.
ii) If $d = n-1$, then

$$N(\lambda, \Omega) = \varphi(\lambda) + O(\lambda^{d/2} \ln(\lambda)) \tag{2.6}$$

when $\lambda \to \infty$.

The next result deals with the degenerate case when $M^*_{Int}(\partial\Omega, d) = +\infty$:

Corollary 2.3. *Let $\Omega \in \mathbb{R}^n$ be an open, bounded set. Let $d \in [n-1, n]$ be the interior Minkowski dimension of $\partial\Omega$. Then we have the following remainder estimates*

$$N(\lambda, \Omega) = \varphi(\lambda) + o(\lambda^{D/2}) \tag{2.7}$$

when $\lambda \to \infty$, for all $D > d$.

Remark 2.4. The same asymtotics are valid changing the Neumann boundary condition by a mixed Dirichlet-Neumann boundary condition, i.e., imposing $u = 0$ in $\Gamma \subset \partial\Omega$ and $\frac{\partial u}{\partial \eta} = 0$ in $\partial\Omega \setminus \Gamma$.

Remark 2.5. When $d = n$ and $M^*_{Int}(\partial\Omega, d) = +\infty$, the Weyl's term $\varphi(\lambda)$ may be changed. See [Pi] for related examples.

3. Proof of the main theorem

We will need in the proof the following tauberian theorem (we refer the reader to [Gui] in [Ho2] for a proof):

Theorem 3.1. *(Hormander) Let $\gamma_n : [0, \infty) \to \mathbb{R}$ be the function $\gamma_n(\lambda) = \lambda^n$ and let $m : [0, \infty) \to \mathbb{R}$ be any non-decreasing function of polynomial growth with $m(0) = 0$. Suppose the cosine transforms of $\frac{dm}{d\lambda}$, $\frac{d\gamma_n}{d\lambda}$ are equal on the interval $|t| \le \delta$. Then*

$$|m(\lambda) - \lambda^n| \le C_n \left(\frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta} \right) \tag{3.1}$$

for $\lambda > 0$, where C_n is a universal constant depending only on n.

Let $\varphi_1, \varphi_2, \dots$ be the normalized eigenfunctions corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots$. Let $E(\lambda)$ be the orthogonal projection on the space spanned by the eigenfunctions corresponding to $\lambda_i \le \lambda^2$. The Schwartz kernel of $E(\lambda)$ is the spectral function

$$e(x, y; \lambda) = \sum_{\lambda_i \le \lambda^2} \varphi_i(x)\varphi_i(y). \tag{3.2}$$

Obviously, $N(\lambda^2) = \int e(x, x; \lambda)dx$, and it is easy to see, using the maximum principle for parabolic equations, that $e(x, x; \lambda) \le C\lambda^n \ \forall x \in \Omega$ (see [Gui] for a proof).

Now we prove Theorem 2.2.

Proof. Following Hormander, the cosine transform

$$u(x, y; t) = \int_0^\infty \cos(\lambda t) \frac{d}{d\lambda} e(x, y; \lambda) d\lambda \tag{3.3}$$

is the fundamental solution of the wave equation (see [R-T] for arbitrary open sets):

$$\begin{cases} \frac{\partial}{\partial t^2} u(x, y; t) = & \Delta u(x, y; t) & \text{in } \Omega \\ \\ \frac{\partial u(x,y;t)}{\partial \eta} = & 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

The free space solution $u_0(x, y; t)$ of 3.4 has a closed form expression,

$$u_o(x, y; t) = \frac{\omega_n}{(2\pi)^n} \int_0^\infty \cos(st) \frac{d}{ds} s^n ds \tag{3.5}$$

where $\omega_n = vol(S^{n-1})/n$, and they must agree with the solution $u(x, y; t)$ if $t < dist(y, \partial\Omega)$, as a consequence of the wave's finite speed of propagation.

Using the Tauberian Theorem (3.1), we obtain:

$$|e(x, x; \lambda) - \frac{\omega_n}{(2\pi)^n} \lambda^n| \le C_n \left(\frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta} \right) \tag{3.6}$$

for $dist(x, \partial\Omega) > \delta$. We can estimate $|N(\lambda) - \frac{\omega_n}{(2\pi)^n}|\Omega|\lambda^n|$ by

$$\int_\Omega |e(x,x;\lambda) - \frac{\omega_n \lambda^n}{(2\pi)^n} dx| \le \int_{\Omega\setminus U} |e(x,x;\lambda) - \frac{\omega_n\lambda^n}{(2\pi)^n}|dx + \int_U C_n \left(\frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta}\right) dx$$

where $U = \{x \in \Omega : dist(x, \partial\Omega) > 1/\lambda\}$.

The second integral is majorized by

$$C\int_{1/\lambda}^{t} \left(\frac{1}{\delta^n} + \frac{\lambda^{n-1}}{\delta}\right) d\delta \tag{3.7}$$

for t large enough, which is of order $O(\lambda^{n-1}\ln(\lambda))$.

To estimate the other integral, we note that, when $\partial\Omega$ is sufficiently smooth, it is of order $O(\lambda^{n-1})$ because we have $e(x,x;\lambda) \le C\lambda^n$ uniformly on Ω, and $|\Omega\setminus U|_n$ is approximately $1/\lambda$ times the perimeter.

In our situation, we can use the interior Minkowski content of $\partial\Omega$ to obtain:

$$|\Omega\setminus U|_n = |\partial\Omega_{1/\lambda} \cap \Omega|_n \le A\left(\frac{1}{\lambda^{n-d}}\right) \tag{3.8}$$

with $A = sup_{\{\varepsilon<1\}}\varepsilon^{-(n-d)}|\partial\Omega_\varepsilon \cap \Omega| < \infty$. It follows immediately that the integral is of order $O(\lambda^d)$, which is greater than the other integral if and only if $d > n-1$. □

Remark 3.2. Note that Corollary (2.3) follows from Theorem (2.2), since according to the definition of Minkowski dimension, $D > d$ implies $M^*_{Int}(\partial\Omega, d) < +\infty$.

Taking d' with $d < d' < D$, we can replace the estimate $O(\lambda^{d'/2})$ obtained from Theorem (2.2) by $o(\lambda^{D/2})$. Further, when $d = n-1$ we can substitute $O(\lambda^{d'/2}\ln(\lambda))$ by $o(\lambda^{D/2})$.

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