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"Capital Accumulation in an Economy with Private Information and Many Heterogeneous Agents"

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1 Introduction

This paper studies capital accumulation in an economy with private information and a …-nite number of heterogeneous agents. The economy is populated by N heterogeneous, risk adverse, in…nitely-lived agents. Agents are endowed with a neoclassical technology which is subject to idiosyncratic productivity shocks. These shocks are assumed to be independently and identically distributed through time and independent across agents. Also, at every date t, the history of realizations up to that date are private information for each agent. Since each individual's stock of capital is observable, it is also assumed that individual's consumption is not.

The economy presented here can be interpreted as a version of the neoclassical growth model with private information and many (but …nite) heterogeneous agents. A standard but apparently implausible prediction of the standard neoclassical growth model with dynamically complete markets is the extreme level of risk-sharing. Recent attempts have arbitrary closed some markets to analyze how predictions would change (see, for example, Aiyagari [2] and Hugget [11]). One of the standard arguments to justify di¤erent incomplete market structures is the fact that there are informational problems and therefore some markets will not be present (see for example Arrow [4]).

The main goal of this paper is to characterize the set of constrained e¢cient allocations in a particular informationally-constrained environment instead of specifying an arbitrary set of insurance markets. That is, as motivated by Townsend [20] and many others, I analyze Pareto optimal arrangements "to avoid the imposition of exogenous restrictions and so the nonexecution of some mutually perceived advantageous trade". Agents will be asked to report their own productivity shocks every period and there is no way to audit or verify the answer any agent chooses to give. I will characterize incentive compatible allocations since a version of the Revelation Principle holds for this economy.

Since there is private information, it is well-known since Townsend [20] that the relevant set of (constrained) e¢cient allocations can be history dependent. Hence, standard recursive methods to characterize optimal allocations do not apply. This can be solved extending the set of state variables to include next period's "discounted expected utility entitlements" as in Abreu et. al. [1] and many others[1] . I will establish the existence of a solution and some properties of the e¢cient allocation. In particular, I will show that any agent must be getting some insurance when the e¢cient allocation is considered and also that the level of discounted expected utility cannot go to zero with positive probability.

The …rst of these results extends Townsend [20]'s seminal contribution to an economy with private information, capital accumulation and an arbitrary …nite number of heterogeneous agents. Townsend [20] introduces the idea that the motive of multiperiod contracts ("enduring relationships") is that agents attempt to circumvent the incentive information dif…culties of single-period agreements. The fact that e¢ciency would imply some risk-sharing is not very surprising. After all, agents can transfer consumption across time and states of natures through capital accumulation. However, agents are not restricted to get insurance only through capital in this paper (in particular, an agent can receive a negative net transfer from the other agents in the economy). This can be contrasted with market economies where insurance markets are assumed away and consumption smoothing is carried out only through

[1] Very important papers are also Spear and Strivastava [18] and Phelan and Townsend [17].

capital accumulation (see, for example, Hugget [11] and Becker and Zilcha [8]).

The second result, initially discussed by Espino [9] for an endowment economy, provides a remarkably di¤erent prediction with respect to a standard result in economies with full information. There, if the marginal utility of consumption goes to in…nity when consumption goes to zero, impatient agents will end up consuming nothing in the limit (this property is brie‡y described below). A novel property of the model presented here is that the introduction of any degree of private information (that is, even if probabilities di¤ers from 1 by an arbitrary small number) will imply that this result will no longer hold.

This paper is organized as follows. Section 2 introduces the original resource allocation problem with private information. Also, some basic properties are established. Section 3 proves the existence of an e¢cient allocation and describes some properties of the e¢cient allocation. Section 4 concludes. The Appendix contains all the proofs.

The remanding of this introduction deals with the relationship of this paper to some of the existing literature.

1.1 The Related Work of Others

Becker [7] studies the long-run behavior of a deterministic economy with many heterogeneous agents. He shows, in particular, that if discount factors are di¤erent across agents, the most patient agent owns, in the limit, all the stock of capital in the economy. Becker and Zilcha [8], however, show that this result does not carry over to stationary stochastic environments where markets are incomplete.

Lucas and Stokey [13] study Pareto optimal allocations in a related deterministic economy where agents' preferences are not necessarily additively separable over time. They characterize optimal allocations through recursive methods using utility entitlements as an additional state variable.

Aiyagari [2] extends the standard growth model to include uninsured idiosyncratic risk and borrowing constraints. The economy is populated by a continuum of ex-ante identical agents. When compared with the complete markets economy, he shows that in the economy he analyses agents overaccumulate capital in order to smooth consumption in the face of idiosyncratic risk. See also Hugget [11] for a related result.

This paper complements the work just described. The economy presented here is a simpli...ed stochastic version of that presented in [13] but which also includes private information. The economies described in [11] and [8] rule out insurance markets where consumption smoothing is carried out by capital accumulation. In Proposition 3, I show that risk-sharing is provided without restricting transfers to be equal to the individual's next period stock of capital. It might perfectly happen that in a market economy where all insurance markets are closed, there are nonexecuted mutually bene...cial trade opportunities. This issue has already been discussed and quantitatively tested by Aiyagari [2]. In one of his examples, by optimally accumulating/decumulating assets, an individual can cut consumption variability by about half and enjoy a welfare gain of about 14% of per capita consumption.

The model with private information originally presented in Townsend [20] was extended by Green [10] and Atkeson & Lucas [6], among others Their economies are populated by a large number of ex-ante identical in...nitely-lived agents. Agents privately observe idiosyncratic shocks. Green assumes that the principal (a ...nancial intermediary) has access to borrowing and lending at a given interest rate. Atkeson and Lucas impose period-by-period feasibility. Independently of the feasibility technologies, one of the striking predictions of their model is the extreme level of "immiserization": the expected utility level of (almost) every agent in the economy converges to the lower bound with probability one. This result is also present in Thomas and Worral [19].

In a related paper, Espino [9] analyzes the interaction between private information and enforceability issues in an endowment economy. His economy is populated by a ...nite number of heterogeneous agents and he assumes that the enforceability in the economy is incomplete. There, agents can exit from contracts with a positive fraction of their endowment when they ...nd it optimal. He shows that "Townsend's long-term relationship property" holds and the expected utility level of no agent converges with positive probability. See also Wang [21] for a related previous result for a simpler economy.

This paper extends the environment described in [9] to allow for capital accumulation in an economy with perfect enforceability. I will show that the basic results presented there will still hold. This is not, however, the ...rst paper to study the interaction between capital accumulation and private information.

Atkeson [5] examines constrained e¢cient allocations between a risk-adverse borrower and a sequence of risk-neutral lenders in an economy where there are two impediments to form contracts. The ...rst one is that borrower's investment (and also consumption) is unobservable to the lenders. This leads to moral hazard problems in investment. The second impediment arises from the assumption that the borrower may choose to repudiate his debts. That is, not all the incentive compatible allocations are enforceable. In this environment, one of his main goals is to show that "capital out‡ows" could be a necessary part of the optimal contract when a low realization of output is observed.

Marcet and Marimon [14] introduce capital accumulation to study the e¤ect on growth of alternative …nancing opportunities in a stochastic growth model with incentive constraints. They characterize constrained e¢cient allocations in an economy with a risk-neutral principal and a risk-adverse agent. In one of the economies they analyze, investment made by the agent is unobservable[2]. They mainly concentrate in the transitional dynamic of the incentive compatible recursive contract. In this partial equilibrium framework, they …nd that information constraints a¤ect consumption volatility while the Pareto e¢cient capital accumulation path is decentralized.

Aiyagari and Williamson [3] include capital accumulation to study the allocation of credit in a random matching model with capacity constraints. Agents have private information about their idiosyncratic endowment shocks. Their main focus is to study e¢cient credit arrangements and then they assume that capital is accumulated by the planner.

Finally, in a closely related unpublished paper, Khan and Ravikumar [12] introduce capital accumulation in Green's model where period-by-period resource constraints are imposed. There is a continuum of agents where each of them is endowed with a linear technology which is subject to idiosyncratic productivity shocks. These shocks are assumed to be private information. Given that there is a large number of agents, agents share risk with …nancial intermediaries through long-term independent contracts. After establishing a recursive formulation of the original problem and a duality property in the spirit of Green [10], they establish through numerical exercises (for CRRA momentary utility functions) that both the

[2] The other constrained economy considered there is one where there is incomplete degree of enforceability: the agent can revert to autarky for ever whenever she …nds it optimal.

expected valued and the standard deviation of utility entitlements grow through time.

In the economy presented in this paper, I will also try to complement some of these results. In particular, I will purposely not consider commitment issues to isolate the effects of the interaction of private information and capital accumulation. I will not consider long-run growth either. This allows showing that the utility possibility set is compact. When this result is complemented with the fact that optimally all agents must be getting some insurance, it follows that, for any agent, the level of expected discounted utility cannot converge. Very importantly, agents are not necessarily ex-ante identical and thus they can differ, in particular, in their discount factors.

2 The Economy

The economy is populated by a finite number of infinitely-lived heterogeneous agents with names in the set $I = \{1, ..., N\}$ (with typical element n). Time is discrete and denoted $t = 0, 1, 2, ...$. There is only one consumption good. Each agent is endowed with a neoclassical technology, $f(k; \mu)$, which is subject to idiosyncratic productivity shocks. That is, if agent n's stock of capital at date t is k_{nt} and he has received a productivity shock μ_{nt}, he can produce $f(k_{nt}; \mu_{nt})$ units of the consumption good. Agents are endowed at date 0 with $k_0 = (k_{01}, ..., k_{0N})$ units of capital. I will assume that, for all $n \in I$ and for all t, $\mu_n^t = \{\mu_{n0}, ..., \mu_{nt}\}$ is private information. Assume that $\mu_{nt} \in \Theta = \{\mu_1, ..., \mu_J\}$ for all $n \in I$ and all $t \geq 0$. This set is ordered: $\mu_j < \mu_{j'}$ if $j < j'$.

Assumption 1: $f : R_+ \times \Theta \to R_+$ is strictly increasing in both arguments, $f(\cdot; \mu_n)$ is strictly concave for all μ_n. There exists $\overline{K} > 0$ such that if $k \leq \overline{K}$ then $Nf(k; \mu_J) \leq \overline{K}$.

Note that this assumption implies that if $\mu_j < \mu_{j'}$ then $f(0; \mu_j) < f(0; \mu_{j'})$. That is, even with no capital any agent can produce a positive amount of the consumption good; in this case production is also increasing in the productivity shocks.

Productivity shocks are assumed to be independent across agents and i.i.d. across time for every agent. That is, let $\pi_{nt}(\theta_j) = \pi_n(\theta_j) > 0$ be the probability for agent n of having a productivity shock θ_j for all $t \geq 0$. Note that there is aggregate uncertainty given that the number of agents is finite. Let $\theta = (\theta_1; \ldots; \theta_N) \in \Theta^N$ denote the aggregate productivity shock with probability $\pi(\theta) = \prod_{n \in N} \pi_n(\theta_n)$. Let μ^{t+1} be the probability distribution on the measurable space $(2^{\Sigma^{N(t+1)}}; \Theta^{N(t+1)})$ induced by π. That is, $\mu^{t+1}(\theta^t)$ is the probability of the aggregate partial history up to date t; $\theta^t = (\theta_0; \ldots; \theta_t) \in \Theta^{N(t+1)}$.

Preferences

Let S denote the consumption set, which is defined in the following way:

$$S = \{\{c_t\}_{t=0}^{\infty} : c_t : \Theta^{N(t+1)} \to R_+ \text{ and } \sup_{t;\theta^t} c_t(\theta^t) < \infty\}$$

Preferences over S are represented by time-separable expected discounted utility; that is, if $c \in S$

$$U(c) = E_0\{\sum_{t=0}^{\infty} \beta_n^t u_n(c_t)\}$$

It is assumed that for all $n \in I$; $\beta_n \in (0;1)$ and $u_n : R_+ \to R$ is strictly increasing and strictly concave; assume also that $\lim_{c_t \to 0} u_n'(c_t) = +\infty$ for all $n \in I$. Without loss of generality, assume $u_n(0) = 0$ for all n. E_0 means, as usual, the expectation operator.

Incentive Compatibility

Since it is assumed that θ_{nt} and θ_n^t are private information, agents will be asked to report their productivity shocks. I will assume that there is no way to audit or verify the answer any agent chooses to give.

Given a partial history θ_n^t up to date t privately observed by agent n, he will report, at date t, $z_{nt}(\theta_n^t) \in \Theta$. Let $z_n = \{z_{nt}(\theta_n^t)\}_{t=0}^{\infty}$ represent agent n's sequence of reporting strategies where $z_{nt} : \Theta^{t+1} \to \Theta$ for all t. Denote $z = (z_1, ..., z_N) = (z_n, z_{-n})$ as the sequence of aggregate reporting strategies. Note that since each individual only observes her own productivity shock, agent n's reporting strategy depends only upon her own partial history.

Let $K' = \{k_{t+1}\}_{t=0}^{\infty}$ be an investment rule where, for all t, $k_{t+1} : \Theta^{N(t+1)} \to R_+^N$. Similarly, let $B = \{B_{nt}\}_{t=0}^{\infty}$ be a net transfer mechanism where, for all t, $B_t : \Theta^{N(t+1)} \to R^N$. To interpret this, consider any aggregate realization θ^t up to date t and any aggregate reporting strategy z. Consumption for each agent n is given by $c_{nt}(z^t(\theta^t)) = f(k_{nt}(z^{t-1}(\theta^{t-1}), \theta_{nt}) + B_{nt}(z^t(\theta^t))$.

The set of allocations and reporting strategies to be considered will be appropiately restricted. Let z^* be the truthtelling reporting strategy where $z_t^*(\theta^t) = \theta_t$ for all t and for all θ^t. A vector of sequences (B, K') is called an allocation if for all t and for all θ^t, $f(k_{nt}(\theta^{t-1}), \theta_{nt}) + B_{nt}(\theta^t) \geq 0$. This means that when all agents are truthfully reporting, an allocation must give nonnegative consumption to each agent. Given an allocation (B, K'), let $Z(B, K', k)$ be the set of reporting strategies such that $z \in Z(B, K', k)$ if and only if for all t and all θ^t, $f(k_{nt}(z^{t-1}(\theta^{t-1}), \theta_{nt}) + B_{nt}(z^t(\theta^t))) \geq 0$ and $k_0 = k$. Note that, by definition of an allocation, $Z(B, K', k)$ is not empty since z^* is in there.

Definition 1 An allocation (B, K') is feasible given k if for all $z \in Z(B, K', k)$

$$\sum_{n \in N} (k_{nt+1}(z^t(\theta^t)) + B_{nt}(z^t(\theta^t)) \leq 0 \tag{1}$$

for all t and all θ^t and $k_0 = k$.

This just means that aggregate investment is not greater than aggregate savings. Note that $k_{nt+1}(z^t(\theta^t)) = -B_{nt}(z^t(\theta^t))$ for all n, all t and all θ^t is a particular case (in that case, each agent can get some insurance only through capital accumulation).

The levels of capital will be also restricted to those which are sustainable. Given Suppose that $k_{nt}(\theta^{t-1}) \leq \overline{K}$ for all n. Since consumption must be nonnegative, from feasibility and the definition of an allocation we have that for n, for all t and for all report θ^t

$$
\begin{aligned}
0 \leq k_{nt+1}(\theta^t) &\leq \sum_{n\in N} k_{nt+1}(\theta^t) \leq \sum_{n\in N} f(k_{nt}(\theta^{t-1});\theta_{nt}) \\
&\leq \sum_{n\in N} f(k_{nt}(\theta^{t-1});\theta_J) \leq \overline{K}
\end{aligned}
$$

Denote $X \equiv [0;\overline{K}]^N$ as the set of sustainable capital levels. It will be assumed that $k_0 \in X$; therefore, any feasible allocation will necessarily have that $k_{t+1}(\theta^t) \in X$ for all t and for all θ^t.

Suppose that some arbitrary θ^{t-1} has been reported. Let z' be an aggregate continuation reporting strategy from period t onwards. Given an allocation $(B;K')$, define for all $z' \in Z(B;K';k_t(\theta^{t-1}))$ the level of expected discounted utility entitled to agent n at date t as follows:

$$
\begin{aligned}
U_{nt}(B;K';z'\|\theta^{t-1}) &= \sum_{s=0}^{\infty} \beta_n^s \int_{\Theta^{N(s+1)}} \{u_n(f(k_{nt}(\theta^{t-1};z^{s-1}(\theta^{s-1}));\theta_{ns})) + B_{nt}(\theta^{t-1};z^s(\theta^s)))\, d\pi^{s+1}(\theta^s)\} \\
&= \sum_{\theta\in\Theta^N} \pi(\theta)\{u_n(f(k_{nt}(\theta^{t-1});\theta_n) + B_{nt}(\theta^{t-1};z_0(\theta))) \\
&\quad + \beta_n U_{nt+1}(B;K';z'(\theta)\|\theta^{t-1};\theta)\}
\end{aligned}
$$

where $z'(\theta)$ is the continuation reporting strategy induced by z' from period $t+1$ onwards when the first element θ is kept constant. When $t = 0$, we write for any $z \in Z(B;K';k)$

$$U_n(B;K';z) = U_{n0}(B;K';z)$$

Let $Z_n(B; K'; k) = \{z_n : (z_n; z^*_{-n}) \in Z(B; K'; k)\}$. The following Lemma will be useful to establish some results "in the limit".

Lemma 1 Let $(B; K')$ be any feasible allocation at $k \in X$. Consider any agent $n \in I$ and let $z'_n; z'^m_n \in Z_n(B; K'; k)$ be continuation reporting strategies where $z^m_{nt} = z_{nt}$ for all $m \leq t$ and $z^m_{nt} = z^*_{nt}$ thereafter. Then, for all $t \geq 0$ and any aggregate report μ^{t-1}

$$\limsup_{m \to \infty} U_{nt}(B; K'; z'^m_n; z^* \| \mu^{t-1}) = U_{nt}(B; K'; z'_n; z^* \| \mu^{t-1})$$

The following definition says that an allocation is incentive compatible if truthtelling is the best response for each agent whenever the other agents are truthfully reporting their own productivity shocks not only today but also in the future. Note that it is taken into the account that agents can choose a continuation reporting strategy every period after they have observed their own productivity shock histories. Note also that the restriction of analyzing incentive compatible through Nash Implementation is without loss of generality since it can be shown that the relevant version of the celebrated Revelation Principle holds (more precisely, it is a well-known result that the revelation principle holds for any time horizon and any stochastic structure). Roughly speaking, if there is any way in which some insurance can be provided through any allocation then there is an equivalent incentive compatible way in which agents report their true productivity shocks.

Definition 2 Given $k_0 \in X$; an allocation $(B; K')$ is incentive compatible if for all agent $n \in I$, for all $t \geq 0$, all μ^{t-1}; $z'_n \in Z_n(B; K'; k_{t+1}(\mu^{t-1}; \mu))$

$$\begin{aligned} & u_n(f(k_{nt}(\mu^{t-1}); \mu_n) + B_{nt}(\mu^{t-1}; \mu_n; \mu_{-n})) + \beta_n U_{nt+1}(B; K'; z^*_n; z^*_{-n} \| \mu^{t-1}; \mu_n; \mu_{-n}) \quad (2) \\ \geq\ & u_n(f(k_{nt}(\mu^{t-1}); \mu_n) + B_{nt}(\mu^{t-1}; \hat{\mu}_n; \mu_{-n})) + \beta_n U_{nt+1}(B; K'; z'_n; z^*_{-n} \| \mu^{t-1}; \hat{\mu}_n; \mu_{-n}) \end{aligned}$$

for all $\hat{\mu}_n; \mu_{-n}$.

The notion of efficiency which will be discussed throughout the paper can now be defined.

Definition 3 An allocation $(B^*; K'^*)$ is efficient at $(k_0; \{u_n\}^N_{n=2})$ if

$$(B^*; K'^*) \in \arg\max_{(B;K')} \{U_1(B; K'; z^*) : (B; K') \text{ satisfies (1)-(2) and}$$
$$U_n(B; K'; z^*) = u_n \text{ for all } n = 2; \ldots; N\}$$

Note that in the definition we are already using the fact that the allocation must be incentive compatible. Also, efficiency does not necessarily mean the feasibility constraint will bind. Even though preferences are assumed to be strongly monotone, consumption cannot be arbitrarily manipulated since incentive compatibility must hold.

Let $\Psi(k)$ be the utility possibility set for this economy when $k \in X$ is the initial stock (and distribution) of capital; that is

$$\Psi(k) \equiv \{u \in R^N : \text{there exists } (B; K^0) \text{ satisfying } (1) - (2) \text{ and all for } n,\ u_n = U_n(B; K^0; z^*)\}$$

This correspondence, mapping X into R^N; has some properties that can be established immediately.

Remark 1 For all $k \in X$; $\Psi(k)$ is nonempty since $(B; K^0) = (0; 0)$ is clearly a feasible and incentive compatible allocation. That is, if we define $u_n = \sum_{\mu_n \in \Theta} \pi_n(\mu_n) u_n(f(k_{n0}; \mu_n))$, then $(u_n)_{n \in N} \in \Psi(k)$:

Remark 2 There exists $u_1 > 0$ such that $u = (u_1; 0; \ldots 0) \in \Psi$ for all $k \in X$: To see this, consider the following allocation: given $K_0 \in X$; define for all for all $t \geq 0$ and all μ^t

$$\begin{aligned} B_{nt}(\mu^t) &= -f(K_{nt}; \mu_{nt}) \text{ for all } n \geq 2 \\ B_{1t}(\mu^t) &= -\sum_{n \geq 2} B_{nt}(\mu^t) \\ K_{nt+1}(\mu^t) &= 0 \text{ for } n \in I \end{aligned}$$

Note that $(B; K^0)$ is an incentive compatible feasible allocation given the definitions of an allocation, reporting strategies and feasibility. Note also that $U_n(B; K^0) = 0$ for all $n \geq 2$ and $U_1(B; K^0) > 0$ given that $f(k; \mu_j) > 0$ for all k:

Remark 3 It is uniformly bounded. That is, there exists a bounded subset of R^N; H, such that for all $k \in X$; $\Psi(k) \subset H$: To see this, note that for all agent n, $0 \leq c_{nt} \leq \sum_{n \in N} f(k_{nt}; \mu_{nt}) \leq N f(\bar{k}; \bar{\mu})$: Therefore, for any $k \in X$ if $u \in \Psi(k)$, then $0 \leq u_n \leq \frac{u_n(N f(\bar{k}; \bar{\mu}))}{1 - \beta_n}$:

2.1 The Full Information Case

I will briefly present a very standard property present in the model described above with full information. That is, I will consider the allocation problem described but without considering the incentive compatibility constraints. Since it is easy to establish that both the First and Second Welfare Theorems hold, the property described below will also hold in a complete markets economy.

Consider the following problem. The planner chooses an allocation$(B; K^0)$ to maximize

$$U_1(B;K^0) = \sum_{t=0} \beta_1^t \left\{ \int_{\Sigma^{N(t+1)}} u_1(f(k_{1t}(\theta^{t-1}); \theta_{1t})) + B_{1t}(\theta^t))\, \pi^{t+1}(\theta^t) \right\}$$

subject to (1) and

$$U_n(B;K^0) = \sum_{t=0} \beta_n^t \left\{ \int_{\Sigma^{N(t+1)}} u_n(f(k_{nt}(\theta^{t-1}); \theta_{nt})) + B_{nt}(\theta^t))\, \pi^{t+1}(\theta^t) \right\} = u_n$$

Note that nonnegativity of consumption is implicit in the definition of an allocation

Suppose that $\beta_1 > \beta_n$ for all $n = 1$: Necessary first order conditions (for the unique interior solution) will imply that for all $n \in I$; for all t and for all θ^t

$$B_{nt}(\theta^t): \quad \alpha_n \beta_n^t u_n'(c_{nt}(\theta^t))\, \pi^{t+1}(\theta^t) = \lambda_t(\theta^t)$$

Here, $\alpha_1 = 1$; $\alpha_n > 0$ is the Lagrange multiplier corresponding to agent n and $\lambda_t(\theta^t) > 0$ is the Lagrange multiplier corresponding to the feasibility constraint at period t if θ^t is the aggregate partial history. Note that for all t and for all θ^t

$$\frac{\beta_1^t u_1'(c_{1t}(\theta^t))}{\alpha_n \beta_n^t u_n'(c_{nt}(\theta^t))} = 1 \tag{3}$$

Since $\beta_1^t / \beta_n^t \to +\infty$; it follows from (3) that as t goes to infinity

$$u_1'(c_{1t}(\theta^t)) / u_n'(c_{nt}(\theta^t)) \to 0$$

Given that consumption is uniformly bounded from above, it is clear then that $c_{nt}(\theta^t) \to 0$ for all $\{\theta_t\}_{t=0}^{\infty}$: Therefore, independently of who is the owner of the stock of capital in the economy, only the patient agent consumes in the limit. One of the main results in what follows is that the introduction of any degree of private information will make this result no longer hold.

3 Characterization

In this section I will first characterize the correspondence defined by Ψ. After that, I will establish some important properties of an efficient allocation.

Let $W : X \to R^N$ be a nonempty, uniformly bounded correspondence. Let $(b; k')$ be a vector-valued function where $b : \Theta^N \to R^N$ and $k' : \Theta^N \to X$. Given any two functions $(b; k')$, we say that the function $w : \Theta^N \to R^N$ is a continuation value function with respect to W if for all $\theta \in \Theta^N$ $w(\theta) \in W(k'(\theta))$. Call $(b; k'; w)$ a recursive allocation.

Definition 4 Given a correspondence W as before, a recursive allocation $(b; k'; w)$ is admissible with respect to W at $k \in X$ if

(1) w is a continuation value function with respect to W.

(2) $(b; k'; w)$ satisfies

(2.a) For all $\theta \in \Theta^N$; $f(k_n; \theta_n) + b_n(\theta) \geq 0$

(2.b) For all $\theta \in \Theta^N$; $\sum_n \{b_n(\theta) + k'_n(\theta)\} \leq 0$

(2.c) Temporary Incentive Compatibility (t.i.c.): For all $n \in I$; for all $\theta_{-n} \in \Theta^{N-1}$; and for all $\theta_n; \bar{\theta}_n \in \Theta$

$$
\begin{aligned}
& u_n(f(k_n; \theta_n) + b_n(\theta_n; \theta_{-n})) + \beta_n w_n(\theta_n; \theta_{-n}) \\
\geq\ & u_n(f(k_n; \theta_n) + b_n(\bar{\theta}_n; \theta_{-n})) + \beta_n w_n(\bar{\theta}_n; \theta_{-n})
\end{aligned}
$$

Let $(b; k'; w)$ be admissible with respect to W at $k \in X$; define for all $n \in I$

$$
e_n(b; k'; w) = \sum_{\theta \in \Theta^N} \pi(\theta) \{u_n(f(k_n; \theta_n) + b_n(\theta)) + \beta_n w_n(\theta)\}
$$

Define, given $k \in X$; the following operator:

$$
\Phi(W; k) \equiv \{(e_n(b; k'; w))_{n \in N} \in R^N : \text{ there exists } (b; k'; w) \text{ admissible with respect to } W \text{ at } k\}
$$

Clearly, this operator maps the set of uniformly bounded correspondences into themselves. The following de...nition extends to correspondences some de...nitions given by Abreu et. al. [1] for sets (a similar previous extension was made by Atkeson [5]).

De...nition 5 A correspondence W : X ! R^N is self-generating if it is nonempty and for all k 2 X; W(k) ½ Φ(W; k):

Proposition 1 Let W be a uniformly bounded and self-generating correspondence. Then, for all k 2 X

$$\Phi(W; k) ½ \Psi(k)$$

The next result establish that Ψ itself is self-generating.

Proposition 2 For all k 2 X, Ψ(k) ½ Φ(Ψ; k):

Note that this implies that $\Phi(\Psi; k) = \Psi(k)$: This recursive representation of the problem turns out to be extremely important to establish the existence of an e¢cient allocation and its properties.

3.1 Existence of an E¢cient Allocation

I will proceed to show that for every k 2 X, there exists an e¢cient allocation as de...ned before. To do that, a few properties of the operator Φ need to be shown. De...ne the graph of a correspondence W : X ! R^N by the following set:

$$\text{graph}(W) = f(w; k)\ 2\ R^N\ £\ X : w\ 2\ W(k)g$$

The next Lemma shows that the operator Φ preserves compactness.

Lemma 2 If graph(W) is compact, then graph(Φ(W)) is also compact.

Remark 4 Note that if graph(W_1) ½ graph(W_2); then graph($\Phi(W_1)$) ½ graph($\Phi(W_2)$): This follows directly from the de...nition of the operator Φ:

The last result we need to show the existence of an e¢cient allocation is the following.

Lemma 3 Ψ has a compact graph.

Given that Ψ has a compact graph, it follows that for all $k \in X$, $\Psi(k)$ is a compact subset of R^N. We need to introduce some notation. Let $\Psi_{-1}(k) \equiv \{u_{-1} = (u_n)_{n=2}^{N} \in R^{N-1} :$ there exists u_1 where $(u_1; u_{-1}) \in \Psi(k)\}$.

For any $k \in X$ and given $u_{-1} \in \Psi_{-1}(k)$, define $\Psi_1(k; u_{-1}) \equiv \{u_1 \in R : (u_1; u_{-1}) \in \Psi(k)\}$. It is clear that for all $k \in X$ and given $u_{-1} \in \Psi_{-1}(k)$, $\Psi_1(k; u_{-1})$ is a compact subset of R. Define, given k and u_{-1}, the following function:

$$U_1^*(k; u_{-1}) = \max\{u_1 \in \Psi_1(k; u_{-1})\}$$

Therefore, it follows that for all $k \in X$ and given $u_{-1} \in \Psi_{-1}(k)$ there exists $(B^*; K'^*)$ such that

$$\begin{aligned} U_1^*(k; u_{-1}) &= U_1(B^*; K'^*; z^*) \\ u_{-1} &= U_{-1}(B^*; K'^*; z^*) \end{aligned}$$

which is, by definition, an efficient allocation at $(k; u_{-1})$. It also follows by Proposition 1 and 2 that there exists an equivalent efficient recursive allocation $(b^*; k'^*; w^*)$ (which is admissible with respect to Ψ at k) such that

$$\begin{aligned} U_1^*(k; u_{-1}) &= e_1(b^*; k'^*; w^*) \\ u_{-1} &= e_{-1}(b^*; k'^*; w^*) \end{aligned}$$

It will be said that a recursive allocation $(b; k'; w)$ is promise keeping at u_{-1} if $u_n = e_n(b; k'; w)$ for all $n \in \{2, ..., N\}$.

3.2 Some Properties of the Efficient Allocation

Some important properties of the efficient allocation (or its equivalent recursive representation) can now be established. Lemma 4 shows that the efficient allocation will display a partial insurance property and that the constraint set can be simplified.

Consider any $j, l \in \{1, ..., J\}$ and $\theta_{-n} \in \Theta^{N-1}$; define for all $n \in I$ and given $k \in X$

$$
\begin{aligned}
l^n_{j,k}(\theta_{-n}, k) &= u_n(f(k_n,\theta_j) + b_n(\theta_j, \theta_{-n})) + \beta_n w_n(\theta_j, \theta_{-n}) \\
&\quad - u_n(f(k_n,\theta_j) + b_n(\theta_k, \theta_{-n})) - \beta_n w_n(\theta_k, \theta_{-n})
\end{aligned}
$$

Lemma 4 Let (b, k', w) be admissible with respect to Ψ at (k, u_{-1}). Then,

(i) For all $n \in I$, if $\theta_n > \hat{\theta}_n$, then for all $\theta_{-n} \in \Theta^{N-1}$

$$
\begin{aligned}
b_n(\theta_n, \theta_{-n}) &\leq b_n(\hat{\theta}_n, \theta_{-n}) \text{ for all } n \in I \\
w_n(\theta_n, \theta_{-n}) &\geq w_n(\hat{\theta}_n, \theta_{-n}) \text{ for } n \in I
\end{aligned}
$$

(ii) For all $n \in I$ and for all $\theta_{-n} \in \Theta^{N-1}$, if for all s

$$l^n_{s,s-1}(\theta_{-n}, k) \geq 0 \text{ and } l^n_{s,s+1}(\theta_{-n}, k) \geq 0$$

Then, $l^n_{j,k}(\theta_{-n}) \geq 0$ for all s and k.

A further characterization of U^* is essential to show the next results. The main properties are summarized in the following Lemma.

Lemma 5 (i) U_1^* is strictly increasing in k and strictly decreasing in u_{-1}.

For some of the following results I will use the following additional assumption.

Assumption 2: U_1^* is a continuous function.

This assumption needs some justification. It is clear that the relevant constraint set defines an upper hemicontinuous correspondence since the graph of Ψ is compact and weak inequalities are preserved in the limit. To apply the Theorem of the Maximum, however, the nontrivial part is to establish that the relevant constraint set defines a lower hemicontinuous correspondence (mainly because $\Psi(k)$ is not necessarily convex for all k). An alternative assumption (but not necessarily weaker) would be to assume that the operator Φ maps convex valued correspondences into themselves. In this case, and along the lines of Lemma 5, one can show that Ψ is a convex-valued correspondence. This last fact and given the de…nition of U_1^{α} and the fact that U_1^{α} is strictly monotone in both arguments, one can then show that the relevant constraint set de…nes a lower hemicontinuous correspondence as in Espino [9]. Instead of making this kind of assumption, and similar to Atkeson [2], I will make the additional Assumption 2 directly.

I present now the last important properties of an e¢cient allocation. First, I will show that an e¢cient allocation must have no agent with consumption being zero if the her productivity shock is the highest one (Lemma 6). Then, Proposition 3 establishes that an allocation providing no insurance for some agent cannot be e¢cient. Insurance is not necessarily obtained through capital accumulation and in particular net transfers are not restricted to be nonnegative. As a matter of fact, an important step in the proof will take as given the capital accumulation path.

Finally, and as a consequence of Proposition 3, I will show that no agent's utility entitlement can converge to any number with positive probability (Proposition 4). In particular, it cannot converge to the lower bound.

Lemma 6 Consider any k 2 X and any u_{i1} 2 $\Psi_{i1}(k)$: If $fb^{\alpha}; k^{0\alpha}; w^{\alpha}g_{\mu 2 \pounds^N}$ is an e¢cient recursive allocation at $(k; u_{i1})$, then for all n 2 I and all μ_{in}

$$f(k_n; \mu_J) + b_n^{\alpha}(\mu_J; \mu_{in}) > 0$$

This result follows basically because of the restrictions imposed by the de…nitions of an allocation and reporting strategies. In particular, given any allocation, no agent can report a productivity shock giving him negative consumption.

Proposition 3 Given any $k\ 2\ X$ and $u_{¡1}\ 2\ \Psi_{¡1}(k)$; consider an arbitrary recursive allocation $(b; k^{0}; w)$ admissible with respect to Ψ at k and promise keeping at $u_{¡1}$. Suppose that for some $n\ 2\ I$ and for all $\mu\ 2\ \Theta^{N}$

$$\begin{aligned} b_n(\mu) &= \bar{b}_n \\ w_n(\mu) &= \overline{w}_n \end{aligned}$$

Then, $(b; k^{0}; w)$ cannot be e¢cient at $(k; u_{¡1})$.

Let $fW_{nt}g_{t=0}^{1}$ be the stochastic process representing agent n^{0}s expected utility entitlement given an e¢cient allocation. Call $\Omega = ff\mu_t g_{t=0}^{1} : \mu_t\ 2\ \Theta^{N}$ for all tg and let $B(\Omega)$ be the Borel ¾ ¡ field of Ω: Let ^{1} be the unique probability measure on $(\Omega; B(\Omega))$ generated by the ...nite-dimensional distributions $(^{1t})$ (as an application of the Kolmogorov's Extension Theorem).

Proposition 4 For any $n\ 2\ I$; any $k\ 2\ X$ and any $\overline{W}_n\ 2\ \Psi_n(k)$;

$$^{1}ff\mu_t g_{t=0}^{1} : \lim_{t!1} W_{nt}(\mu^{t}) = \overline{W}_n g = 0:$$

An important remark must be made here. The introduction of any degree of private information precludes the result described in 2.1. for economies with full information. There, the most patient agent will consume all the output in the limit.

4 Conclusions

This paper has studied some properties of an economy which can be interpreted as a version of the stochastic neoclassical growth model with many heterogeneous agents and private information. To show the existence of an e¢cient allocation, the …rst step was to prove that the original allocation problem had a recursive formulation in the spirit of Abreu et. al [1] and others. Then, basically two main properties of an e¢cient allocation have been established.

First I have shown that every agent must get some insurance whenever the e¢cient allocation is considered. Unlike most of the literature considering an arbitrary set of incomplete markets, agents can make transfers contingent to their own idiosyncratic productivity shocks. The type of analysis developed in this paper avoids the presence of some mutually bene…cial nonexecuted trade opportunities.

Secondly, I have shown that the level of expected discounted utility cannot converge with positive probability to the lower bound. This result shows that a standard property of e¢cient allocations in economies with full information does no longer hold when any degree of private information is considered. In those economies, and under standard assumptions, the impatient agents will end up consuming nothing in the limit and therefore the level of expected utility converges to the lower bound. This result does not hold anymore with private information and it is independent of who is the owner of the aggregate stock of capital in the limit.

Some extensions could be analyzed. In the …rst place, a natural theoretical extension would be to try to characterize in some more detail both the dynamic and the limiting properties of the relevant variables in the economy. However, at the level of generality presented in this paper, this might not be a standard task.

Secondly, one might try to identify an algorithm to compute e¢cient allocations Thus, numerical results could allow to compute, for example, welfare losses imposed by the information structure when compared with e¢cient allocations in economies with full information. Moreover, one could also compare the basic welfare properties of the economy described here with those emerging in economies where di¤erent markets structures are imposed. In general, what I think is one of the most relevant questions in terms of welfare can be summarized as follows: does private information matter? These issues are left for future research.

5 Appendix

Proof of Lemma 1. Consider any $k > t$. Note that since for all $t \geq 0$ and any aggregate report θ^{t-1}

$$U_{nt}(B; K^0; z_n^{0m}; z^* k\theta^{t-1}) = \sum_{\theta \in \Sigma^N} \pi(\theta)\{u_n(f(k_{nt}(\theta^{t-1}); \theta_n) + B_{nt}(\theta^{t-1}; \theta)) + \beta_n U_{nt+1}(B; K^0; z_n^{0m-1}(\theta); z^* k\theta^{t-1}; \theta)\}$$

we have that

$$\left| U_{nt}(B; K^0; z_n^{0m}; z^* k\theta^{t-1}) - U_{nt}(B; K^0; z_n^0; z^* k\theta^{t-1}) \right|$$

$$\leq \beta_n \sup_{\theta} \left| U_{nt+1}(B; K^0; z_n^{0m}(\theta); z^* k\theta^{t-1}; \theta) - U_{nt+1}(B; K^0; z_n^0(\theta); z^* k\theta^{t-1}; \theta) \right|$$

$$\leq \beta_n^m \sup_{\theta^m} \left| U_{nt+1}(B; K^0; z_n^*; z^* k\theta^{t-1}; \theta^m) - U_{nt+1}(B; K^0; z_n^0(\theta^m); z^* k\theta^{t-1}; \theta^m) \right|$$

Since consumption must be uniformly bounded, taking $\limsup_{m \to \infty}$ in the previous expression the desired result is obtained. ■

Proof of Proposition 1. Let $w_0 \in \Phi(W; k)$ for some given $k \in X$. We need to show that there exists a feasible and incentive compatible allocation $(B; K^0)$ such that for $n \in I$

$$U_n(B; K^0; z^*) = w_{n0}$$

Step 1. Since $w_0 \in \Phi(W; k)$; there exists $(b; k^0; w)(w_0)$ admissible with respect to W at k such that $e_n(b; k^0; w)(w_0) = w_0$. Then, because of the definition of admissibility, for all $\theta \in \Theta^N$ $w(\theta)(w_0) \in W(k^0(\theta)) \subset \Phi(W; k^0(\theta))$ (where the last inclusion follows because W is assumed to be self-generating). It is then clear that we can recursively define, for all $t \geq 0$;

for all μ^t and given w_0;

$$k_{t+1}(\mu^t) = k^0(\mu_t)(W_t(\mu^{t\,i\,1}))$$

$$B_t(\mu^t) = b(\mu_t)(W_t(\mu^{t\,i\,1}))$$

$$W_{t+1}(\mu^t) = w(\mu_t)(W_t(\mu^{t\,i\,1}))$$

Note that in the construction of this candidate allocation $(B; K^0)$; we are only considering truthtelling continuation reporting strategies. We claim now that for all n 2 I; for all t ¸ 0 and for all $\mu^{t\,i\,1}$

$$W_{nt}(\mu^{t\,i\,1}) = U_{nt}(B; K^0; z^{\text{¤}}k\mu^{t\,i\,1}) \tag{4}$$

To see this, note that it follows from de…nition that

$$U_{nt}(B; K^0; z^{\text{¤}}k\mu^{t\,i\,1}) = \sum_{\mu 2 \text{£}^N} \text{¼}(\mu) f u_n(f(k_{nt}(\mu^{t\,i\,1}); \mu_n) + B_{nt}(\mu^{t\,i\,1}; \mu)) + {}^{\bar{}}{}_n U_{nt+1}(B; K^0; z^{\text{¤}}k\mu^{t\,i\,1}; \mu) g$$

and

$$W_t(\mu^{t\,i\,1}) = \sum_{\mu 2 \text{£}^N} \text{¼}(\mu) f u_n(f(k_{nt}(\mu^{t\,i\,1}); \mu_n) + B_{nt}(\mu^{t\,i\,1}; \mu)) + + {}^{\bar{}}{}_n W_{t+1}(\mu^{t\,i\,1}; \mu) g$$

Therefore,

$$\bar{\bar{}} W_t(\mu^{t\,i\,1}) \; i \; U_{nt}(B; K^0; z^{\text{¤}}k\mu^{t\,i\,1}) \bar{\bar{}} \cdot {}^{\bar{}}{}_n \sup_{\mu} \bar{\bar{}} W_{t+1}(\mu^{t\,i\,1}; \mu) \; i \; U_{nt+1}(B; K^0; z^{\text{¤}}k\mu^{t\,i\,1}; \mu) \bar{\bar{}}$$

$$\cdot {}^{\bar{}}{}_n^m \sup_{\mu^m} \bar{\bar{}} W_{t+m}(\mu^{t\,i\,1}; \mu^m) \; i \; U_{nt+m}(B; K^0; z^{\text{¤}}k\mu^{t\,i\,1}; \mu^m) \bar{\bar{}}$$

Since W is a uniformly bounded correspondence and $fB_t g_{t=0}^1$ is uniformly bounded by construction, taking the limit as m ! 1 we get (4) as desired ($^{\bar{}}{}_n$ 2 (0; 1) for all n 2 I).

Step 2. We need to show that (B, K') is a feasible and incentive compatible allocation.

(a) Feasibility follows because $(b(\theta_t), k'(\theta_t), w(\theta_t))(W_t(\theta^{t-1}))$ are admissible with respect to W at $K_t(\theta^{t-1}) \in X$.

(b) Incentive compatibility of our candidate allocation will be proved as usual. First, we will prove that it holds for strategies that have a finite number of deviation from truthtelling. It will follow then from Lemma 1 that it cannot be violated by any reporting strategy with infinitely many deviation from truthtelling.

It follows from admissibility, equality (4) and by construction of (B, K') that

$$u_n(f(k_{nt}(\theta^{t-1}), \theta_n) + B_{nt}(\theta^{t-1}, \theta_n, \theta_{-n})) + \beta_n U_{nt+1}(B, K', z^*|\theta^{t-1}, \theta_n, \theta_{-n}) \quad (5)$$
$$\geq u_n(f(k_{nt}(\theta^{t-1}), \theta_n) + B_{nt}(\theta^{t-1}, \hat{\theta}_n, \theta_{-n})) + \beta_n U_{nt+1}(B, K', z^*|\theta^{t-1}, \hat{\theta}_n, \theta_{-n})$$

for all $n \in I$, $t \geq 0$, θ^{t-1}, θ_n and $\hat{\theta}_n$.

Note that this is not completely satisfying (2) since it has to hold for all $z^0_n \in Z_n(B, K', k_{t+1}(\theta^t))$. Let z^m_n be defined as in Lemma 1. We want to show that for all $m \geq 0$

$$u_n(f(k_{nt}(\theta^{t-1}), \theta_n) + B_{nt}(\theta^{t-1}, \theta_n, \theta_{-n})) + \beta_n U_{nt+1}(B, K', z^*|\theta^{t-1}, \theta_n, \theta_{-n}) \quad (6)$$
$$\geq u_n(f(k_{nt}(\theta^{t-1}), \theta_n) + B_{nt}(\theta^{t-1}, \hat{\theta}_n, \theta_{-n})) + \beta_n U_{nt+1}(B, K', z^m_n, z^*_{-n}|\theta^{t-1}, \hat{\theta}_n, \theta_{-n})$$

for all $n \in I$, $t \geq 0$, θ^{t-1}, θ_n and $\hat{\theta}_n$. Note that (6) holds for $m = 0$ since (5) holds. Suppose that (6) holds for some m. Note that for all θ^t

$$\begin{aligned} U_{nt+1}(B, K', z^{m+1}_n, z^*_{-n}|\theta^t) &= \sum_{\theta \in \Theta^N} \pi(\theta)\{u_n(f(k_{nt+1}(\theta^t), \theta_n) + B_{nt}(\theta^t, z^{m+1}_{n0}(\theta_n), \theta_{-n})) \\ &\quad + \beta_n U_{nt+2}(B, K', z^m_n(\theta), z^*_{-n}|\theta^{t-1}, z^{m+1}_{n0}(\theta_n), \theta_{-n})\} \\ &\geq \sum_{\theta \in \Theta^N} \pi(\theta)\{u_n(f(k_{nt+1}(\theta^t), \theta_n) + B_{nt}(\theta^t, \theta_n, \theta_{-n})) \end{aligned}$$

$$+^{-}{}_{n}U_{nt+2}(B; K^{0}; z^{\alpha}k\mu^{t\,i\,1}; \mu_n; \mu_{\,i\,n})g$$
$$= \quad U_{nt+1}(B; K^{0}; z^{\alpha}k\mu^{t})$$

where the ...rst inequality follows because (6) is supposed to hold for m: Hence, given this inequality and (4) we get

$$u_n(f(k_{nt}(\mu^{t\,i\,1}); \mu_n) + B_{nt}(\mu^{t\,i\,1}; \beta_n; \mu_{\,i\,n})) + {}^{-}{}_{n}U_{nt+1}(B; K^{0}; z_n^{m+1}; z^{\alpha}_{\,i\,n}k\mu^{t\,i\,1}; \beta_n; \mu_{\,i\,n})$$
$$\cdot \quad u_n(f(k_{nt}(\mu^{t\,i\,1}); \mu_n) + B_{nt}(\mu^{t\,i\,1}; \beta_n; \mu_{\,i\,n})) + {}^{-}{}_{n}U_{nt+1}(B; K^{0}; z^{\alpha}k\mu^{t\,i\,1}; \beta_n; \mu_{\,i\,n})$$
$$\cdot \quad u_n(f(k_{nt}(\mu^{t\,i\,1}); \mu_n) + B_{nt}(\mu^{t\,i\,1}; \mu_n; \mu_{\,i\,n})) + {}^{-}{}_{n}U_{nt+1}(B; K^{0}; z^{\alpha}k\mu^{t\,i\,1}; \mu_n; \mu_{\,i\,n})$$

It follows by induction that (6) holds for all m ¸ 0: Finally, consider any arbitrary reporting strategy z_n^{0} 2 $Z_n(B; K^{0}; k_{t+1}(\mu^{t}))$ (included those with in...nitely many misreport). If (2) does not hold then it follows by Lemma 1 that it should not hold for some large enough m. But that is a contradiction to (6). ■

Proof of Proposition 2. Given some arbitrary k 2 X; let $\bar{u}$ 2 $\Psi(k)$: Then there exists a feasible and incentive compatible allocation $(B; K^{0})$ such that $\bar{u}_n = U_n(B; K^{0}; z^{\alpha})$: Put for all n 2 I and all μ 2 Θ^{N}

$$b_n(\mu) \quad = \quad B_{n0}(\mu); \quad k_n^{0}(\mu) = K_{n1}(\mu)$$
$$w_n(\mu) \quad = \quad U_{n1}(B; K^{0}; z^{\alpha}k\mu)$$

Note that by construction, $\bar{u} = e_n(b; k^{0}; w)$: We need to check that $(b; k^{0}; w)$ is admissible with respect to Ψ at k: To do so, we will ...rst check that $w(\mu)$ 2 $\Psi(k^{0}(\mu))$ for all μ: Fix an arbitrary $\bar{\mu}$; put for all t ¸ 0 and all μ^{t}

$$\dot{\bar{B}}_t(\mu^{t}) \quad = \quad B_{t+1}(\bar{\mu}; \mu^{t})$$

$$\overline{k}_{t+1}(\mu^t) \;=\; k_{t+2}(\overline{\mu}; \mu^t)$$

Clearly, $(\overline{B}; \overline{K})$ is feasible at $k_1(\overline{\mu})$by construction. Also it is incentive compatible since $(B; K)$ actually is (see condition (2)). Therefore, since $\overline{\mu}$ was arbitrary, we can conclude that Ψ is self-generating.

Since Ψ is uniformly bounded (see Remark (2) above), we can then conclude from Proposition 1 and 2 that for all k 2 X

$$\Psi(k) = \Phi(\Psi; k)$$

■

Proof of Lemma 2. Let $\mathrm{f}u^j; k^j \mathrm{g}_{t=0}^{1}$ be a sequence in graph($\Phi(W)$): Then, for all j there exists $(b^j; k^{0j}; w^j)$ admissible with respect to W at k^j where

$$u^j \;=\; e(b^j; k^{0j}; w^j)$$

$$w^j(\mu) \quad 2 \quad \Psi(k^{0j}(\mu)) \text{ for all } \mu$$

Given that $\mathrm{f}k^j \mathrm{g}_{t=0}^{1}$ ½ X; it has a convergent subsequence with limit $\overline{k}$ 2 X. But then since $\mathrm{f}k^{0j}; w^j \mathrm{g}_{t=0}^{1}$ is a sequence in graph(Ψ); a compact set, it also has a convergent subsequence. Also, by the de...nition of admissibility, $\mathrm{f}b^j\mathrm{g}$ is also in a compact set having then a convergent subsequence (and the limit satis...es all the conditions imposed by admissibility). Therefore, for each μ 2 Θ^N; there exists $(\overline{b}(\mu); \overline{k}^0(\mu); \overline{w}(\mu))$ being the limit point to this convergent subsequence. Clearly, given that momentary utility function are assumed to be continuous and weak inequalities are preserved in the limit, $(\overline{b}(\mu); \overline{k}^0(\mu); \overline{w}(\mu))_{\mu 2 \pounds^N}$ is admissible with respect to $\bar{k}$. Therefore,

$$\lim_{m\to\infty}\left(\sum_{\theta\in\Theta^N}\pi(\theta)\{u_n(f(k_n^{j_m};\theta_n)+b_n^{j_m}(\theta))+\beta_n w_n^{j_m}(\theta)\}\right)_{n\in N}=(e_n(\bar{b};\bar{k}';\bar{w}))_{n\in N}\in\Phi(\Psi;\bar{k})$$

which establishes that graph($\Phi(W)$) is a compact set. ■

Proof of Lemma 3. We already know that graph(Ψ) is a bounded set. We need to show that it is also closed. Define the correspondence $\overline{\Psi}$ such that

$$\text{graph}(\overline{\Psi})=\text{closure}(\text{graph}(\Psi))$$

Clearly, it follows by definition that graph(Ψ) $\subset$ graph($\overline{\Psi}$). By the previous remark, graph($\Phi(\Psi)$) $\subset$ graph($\Phi(\overline{\Psi})$). Since $\Psi=\Phi(\Psi)$, graph($\Phi(\Psi)$) = graph(Ψ) and graph(Ψ) $\subset$ graph($\Phi(\overline{\Psi})$).

Since graph($\overline{\Psi}$) is closed by definition, from Lemma 3 we have that graph($\Phi(\overline{\Psi})$)is also closed. Hence, graph($\overline{\Psi}$) = closure(graph(Ψ)) $\subset$ closure(graph($\Phi(\overline{\Psi})$)) = graph($\Phi(\overline{\Psi})$) and therefore for all $k\in X$, $\overline{\Psi}(k)\subset\Phi(\overline{\Psi};k)$. But then $\overline{\Psi}$ is self-generating and from Proposition 1 we know that $\overline{\Psi}(k)\subset\Psi(k)$ for all $k\in X$. This implies that graph($\overline{\Psi}$) $\subset$ graph(Ψ) and thus graph(Ψ) is closed. ■

Proof of Lemma 4. (i) Given any θ_{-n}, consider $\theta_n>\hat{\theta}_n$ and note that

$$u_n(f(k_n;\theta_n)+b_n(\theta_n;\theta_{-n}))+\beta_n w_n(\theta_n;\theta_{-n})$$
$$\geq u_n(f(k_n;\theta_n)+b_n(\hat{\theta}_n;\theta_{-n}))+\beta_n w_n(\hat{\theta}_n;\theta_{-n})$$

and

$$u_n(f(k_n;\hat{\theta}_n)+b_n(\hat{\theta}_n;\theta_{-n}))+\beta_n w_n(\hat{\theta}_n;\theta_{-n})$$
$$\geq u_n(f(k_n;\hat{\theta}_n)+b_n(\theta_n;\theta_{-n}))+\beta_n w_n(\theta_n;\theta_{-n})$$

imply

$$u_n(f(k_n;\theta_n)+b_n(\theta_n;\theta_{-n})) - u_n(f(k_n;\hat{\theta}_n)+b_n(\theta_n;\theta_{-n}))$$

$$\geq u_n(f(k_n;\theta_n)+b_n(\hat{\theta}_n;\theta_{-n})) - u_n(f(k_n;\hat{\theta}_n)+b_n(\hat{\theta}_n;\theta_{-n}))$$

Since u_n is strictly concave, it follows that $b_n(\theta_n;\theta_{-n}) \leq b_n(\hat{\theta}_n;\theta_{-n})$. From the previous inequalities stated in this proof, it follows that $w_n(\theta_n;\theta_{-n}) \geq w_n(\hat{\theta}_n;\theta_{-n})$.

(ii) This part follows by standard arguments. See, for example, Espino [9]. ∎

Proof of Lemma 5. (i) Suppose that $k_n < \hat{k}_n$ for some n. Fix $(\overline{u}_{-1};k_{-n})$ and assume initially that $n = 1$. The same argument works for $n = 1$.

Consider any recursive efficient allocation $(b^*;k'^*;w^*)$ at $(\overline{u}_{-1};k_n;k_{-n})$ and note that

$$0 \leq c_n^*(\theta) = f(k_n;\theta_n) + b_n^*(\theta_n;\theta_{-n}) < f(\hat{k}_n;\theta_n) + b_n^*(\theta_n;\theta_{-n})$$

since f is strictly increasing. Define $\hat{b}_n$ as follows. For all θ

$$\hat{b}_n(\theta_n;\theta_{-n}) = c_n^*(\theta) - f(\hat{k}_n;\theta_n) < b_n^*(\theta_n;\theta_{-n})$$

Note that, by construction, $(\hat{b}_n;w_n^*)$ is t.i.c. for agent n. I claim that it can be found $\hat{b}_1(\hat{\theta}) > b_1^*(\hat{\theta})$ for some $\hat{\theta}$ such that $((\hat{b}_1;\hat{b}_n;b^*_{-\{1,n\}});k'^*;w^*)$ is admissible with respect to Ψ at $(\overline{u}_{-1};\hat{k}_n;k_{-n})$. Feasibility will be clear since $\hat{b}_n(\theta) < b_n^*(\theta)$ for all θ. To simplify the analysis, suppose that $\Theta = \{\underline{\theta};\overline{\theta}\}$. Fix any $\hat{\theta}_{-1}$ and note that one can manipulate the t.i.c's. to get

$$u_1(f(k_1;\underline{\theta}) + b_1^*(\underline{\theta};\hat{\theta}_{-1})) - u_1(f(k_1;\underline{\theta}) + b_1^*(\overline{\theta};\hat{\theta}_{-1}))$$

$$\geq \beta_1[w_1^*(\overline{\theta};\hat{\theta}_{-1}) - w_1^*(\underline{\theta};\hat{\theta}_{-1})]$$

$$\geq u_1(f(k_1;\overline{\theta}) + b_1^*(\underline{\theta};\hat{\theta}_{-1})) - u_1(f(k_1;\overline{\theta}) + b_1^*(\overline{\theta};\hat{\theta}_{-1}))$$

If $b_1^*(\underline{\mu};\theta_{-1})) = b_1^*(\bar{\mu};\theta_{-1})$; then $w_1^*(\bar{\mu};\theta_{-1}) = w_1^*(\underline{\mu};\theta_{-1})$: In this case, de...ne

$$\hat{b}_1(\underline{\mu};\theta_{-1})) = \hat{b}_1(\bar{\mu};\theta_{-1}) = b_1^*(\underline{\mu};\theta_{-1})) + \varepsilon$$

and it is clear that t.ic. is satis...ed. Choose ε small enough such that feasibility is satis...ed.

If $b_1^*(\underline{\mu};\theta_{-1})) > b_1^*(\bar{\mu};\theta_{-1})$ (the only alternative possibility given Lemma 4), it follows that

$$\begin{aligned} & u_1(f(k_1;\underline{\mu}) + b_1^*(\underline{\mu};\theta_{-1})) - u_1(f(k_1;\underline{\mu}) + b_1^*(\bar{\mu};\theta_{-1})) \\ > \; & u_1(f(k_1;\bar{\mu}) + b_1^*(\underline{\mu};\theta_{-1})) - u_1(f(k_1;\bar{\mu}) + b_1^*(\bar{\mu};\theta_{-1})) \end{aligned}$$

by strict concavity of u_1: Suppose ...rst that

$$\beta_1[w_1^*(\bar{\mu};\theta_{-1}) - w_1^*(\underline{\mu};\theta_{-1})] > u_1(f(k_1;\bar{\mu}) + b_1^*(\underline{\mu};\theta_{-1})) - u_1(f(k_1;\bar{\mu}) + b_1^*(\bar{\mu};\theta_{-1})) \quad (7)$$

Take $\varepsilon > 0$ and de...ne $\hat{b}_1(\underline{\mu};\theta_{-1})) = b_1^*(\underline{\mu};\theta_{-1})) + \varepsilon$; choose ε small enough such that feasibility is satis...ed and (7) is also satis...ed at least with a weak inequality. When

$$u_1(f(k_1;\underline{\mu}) + b_1^*(\underline{\mu};\theta_{-1})) - u_1(f(k_1;\underline{\mu}) + b_1^*(\bar{\mu};\theta_{-1})) > \beta_1[w_1^*(\bar{\mu};\theta_{-1}) - w_1^*(\underline{\mu};\theta_{-1})]$$

the analysis is similar.

We can then conclude that U^* is strictly increasing in k.

A similar analysis can be used to show that U^* is strictly decreasing in u_{-1}: See Espino [9] for related details. ■

Proof of Lemma 6. The proof presented is rather informal. Details are standard and left to the reader. Note ...rst that if $f(k_n;\mu_J) + b_n^*(\mu_J;\mu_{-n}) = 0$, then $f(k_n;\mu_j) + b_n^*(\bar{\mu};\mu_{-n}) < 0$ for all $\mu_j = \mu_J$: Hence, given any μ_{-n}; if the productivity shock for agent n is μ_J and

$f(k_n;\theta_j) + b^*_n(\theta_J;\theta_{-n}) < 0$, the only incentive compatibility constraint binding would be those of the following way:

$$u_n(f(k_n;\theta_J) + b_n(\theta_J;\theta_{-n})) + \beta_n w_n(\theta_J;\theta_{-n})$$

$$\geq u_n(f(k_n;\theta_J) + b_n(\theta_j;\theta_{-n})) + \beta_n w_n(\theta_j;\theta_{-n})$$

for all $\theta_j < \theta_J$. Let $\lambda^n_{\bar\theta;\theta_j}(\theta_{-n})$ be the Lagrange multiplier corresponding to this constraint; similarly, let $\gamma(\theta_J;\theta_{-n})$ and η_n be the Lagrange multipliers corresponding to the feasibility constraint when the aggregate state is $(\theta_J;\theta_{-n})$ and that to the promise keeping constraint for agent n respectively (if $n = 1$, $\eta_1 = 1$). The following first order condition is then necessary

$$\eta_n \pi(\theta_J;\theta_{-n}) u'_n(f(k_n;\theta_J)+b^*_n(\theta_J;\theta_{-n})) + \sum_{\theta_j=\bar\theta} \lambda^n_{\bar\theta;\theta_j}(\theta_{-n}) u'_n(f(k_n;\theta_J)+b^*_n(\theta_J;\theta_{-n})) - \gamma(\theta_J;\theta_{-n}) \leq 0$$

(with equality if $f(k_n;\theta_J)+b^*_n(\bar\theta;\theta_{-n}) > 0$). Since $\eta_n > 0$ and $(\sum_{\theta_j=\bar\theta} \lambda^n_{\bar\theta;\theta_j}(\theta_{-n}); \gamma(\theta_J;\theta_{-n})) \geq 0$; and given that $u'(0) = +\infty$, it follows that $f(k_n;\theta_J) + b^*_n(\theta_J;\theta_{-n}) > 0$. ■

Proof of Proposition 3. Assume, on the contrary, that $(b;k';w)$ is efficient at $(k;u_{-1})$ where $u_n = e_n(b;k';w)$. Without loss of generality, suppose that $n = 1$ (as it will be clear, the whole proof goes through when $n = 1$). Also, assume to simplify that $\Theta = \{\underline{\theta};\bar\theta\}$ where $\underline{\theta} < \bar\theta$. There will be three cases.

Case 1. $\bar w_n > 0$. Fix some $\tilde\theta_{-\{1;n\}}$ and consider the following alternative recursive allocation: define $\hat\theta = (\bar\theta_1;\underline{\theta}_n\tilde\theta_{-\{1;n\}})$ and put

$$\hat b_n(\hat\theta) = \bar b_n + \lambda \text{ and } \hat b_1(\hat\theta) = b_1(\hat\theta) - \lambda$$

$$\hat w_n(\hat\theta) = \bar w_n - \delta_n \text{ and } \hat w_1(\hat\theta) = w_1(\hat\theta) + \delta_1$$

If $\theta = \hat{\theta}$, put $\hat{b}_n(\theta) = \bar{b}_n$, $\hat{b}_1(\theta) = b_1(\theta)$; $\hat{w}_n(\theta) = \bar{w}_n$ and $\hat{w}_1(\theta) = w_1(\theta)$: For all $i \in I \setminus \{1; n\}$; put simply for all θ $\hat{b}_i(\theta) = b_i(\theta)$ and $\hat{w}_i(\theta) = w_i(\theta)$: Finally, let $\hat{k}' = k'$: We will restrict $(\lambda; \delta_1; \delta_n) \gg 0$ such that $(\hat{b}; \hat{k}'; \hat{w})$ is admissible with respect to Ψ at k, promise keeping at u_{-1} and $e_1(\hat{b}; \hat{k}'; \hat{w}) > e_1(b; k'; w)$:

Step 1.1. Note first $(\delta_1; \delta_n)$ can be chosen to be both positive and $(\hat{w}_1(\hat{\theta}); \hat{w}_n(\hat{\theta}); \{\hat{w}_i(\hat{\theta})\}_{i \in I \setminus \{1;n\}}) \in \Psi(k'(\hat{\theta}))$: To see this, we know from Lemma 5 that U_1^* is strictly decreasing and by Assumption 2 U_1^* is continuous; hence, since if $\bar{w}_n - \delta_1 \geq 0$ and $\delta_1 > 0$

$$w_1(\hat{\theta}) \leq U_1^*(\{\bar{w}_n; w_i(\theta)\}_{i \in I \setminus \{1;n\}}; k'(\hat{\theta})) < U_1^*(; \{\bar{w}_n - \delta_n; w_i(\theta)\}_{i \in I \setminus \{1;n\}}; k'(\hat{\theta}))$$

we can find some $\delta_1 > 0$ such that $(\hat{w}_1(\hat{\theta}); \hat{w}_n(\hat{\theta}); \{\hat{w}_i(\hat{\theta})\}_{i \in I \setminus \{1;n\}}) \in \Psi(k'(\hat{\theta}))$:

Step 1.2. Feasibility is satisfied by definition if $\lambda > 0$ is chosen such that

$$f(k_n; \bar{\theta}) + b_n(\hat{\theta}) - \lambda \geq 0$$

This can be done because $f(k_n; \bar{\theta}) + b_n(\hat{\theta}) > 0$ by Lemma 6.

Step 1.3. Incentive Compatibility. Note first that since the recursive allocation $(b; k'; w)$ is assumed to be admissible with respect to Ψ at k; there is nothing to check for agent $i \in I \setminus \{1; n\}$: Also, since when $\theta_{-\{1;n\}} = \hat{\theta}_{-\{1;n\}}$ nothing has been changed for agents 1 and n (with respect to $(b; k'; w)$), incentive compatibility is satisfied by construction. Suppose then that $\hat{\theta}_{-\{1;n\}}$ has been reported.

Consider first agent 1. If agent n has reported $\theta_n = \underline{\theta}$; there is again nothing to check. Suppose then that agent n has reported $\underline{\theta}$: Note first that if $f(k_1; \underline{\theta}) + b_1(\bar{\theta}; \underline{\theta}_n; \hat{\theta}_{-\{1;n\}}) \leq 0$ and $\lambda > 0$; there is only one incentive compatibility to check (that is, when the agent has as a true productivity shock $\bar{\theta}$ and he does not have incentives to report $\underline{\theta}$) given our definition of reporting strategies. To consider the general case, suppose that $f(k_n;\underline{\theta})+b_n(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}}) > 0$: Define

$$g_1(\underline{\theta};\lambda) = u_1(f(k_1;\underline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})) - u_1(f(k_1;\underline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})-\lambda)$$

$$g_1(\overline{\theta};\lambda) = u_1(f(k_1;\overline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})) - u_1(f(k_1;\overline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})-\lambda)$$

Note that $g_1(\underline{\theta};0) = g_1(\overline{\theta};0) = 0$ and both function are strictly increasing whenever $\lambda > 0$ and $f(k_1;\underline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})-\lambda > 0$: More important, since u_1 is assumed to be strictly concave and $f(k_1;\cdot)$ is strictly increasing for all k_1, it follows that if $\lambda > 0$ then $g_1(\underline{\theta};\lambda) > g_1(\overline{\theta};\lambda)$: Hence,

$$\begin{aligned} & u_1(f(k_1;\underline{\theta})+b_1(\underline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}}))+\beta_1 w_1(\underline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}}) \\ \geq\ & u_1(f(k_1;\underline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})-\lambda)+\beta_1[w_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})+\delta_1] \\ =\ & u_1(f(k_1;\underline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})) - g_1(\underline{\theta};\lambda)+\beta_1[w_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})+\delta_1] \end{aligned}$$

is satisfied if and only if

$$l^1_{\underline{\theta};\overline{\theta}}(\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}};k)+g_1(\underline{\theta};\lambda) \geq \beta_1\delta_1 \qquad (8)$$

Also,

$$\begin{aligned} & u_1(f(k_1;\overline{\theta})+b_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})-\lambda)+\beta_1[w_1(\overline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}})+\delta_1] \\ \geq\ & u_1(f(k_1;\overline{\theta})+b_1(\underline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}}))+\beta_1 w_1(\underline{\theta};\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}}) \end{aligned}$$

is satisfied if and only if

$$l^1_{\overline{\theta};\underline{\theta}}(\underline{\theta}_n;\boldsymbol{\theta}_{-\{1,n\}};k)+\beta_1\delta_1 \geq g_1(\overline{\theta};\lambda) \qquad (9)$$

Consider the relevant levels of (λ, δ_1) to put $g_1(\underline{\theta};\lambda) = \beta_1\delta_1 > g_1(\overline{\theta};\lambda)$. Since $I^1_{\underline{\theta},\overline{\theta}}(\theta_n;\hat{\theta}_{-\{1,n\}};k) \geq 0$ and $I^1_{\overline{\theta},\underline{\theta}}(\theta_n;\hat{\theta}_{-\{1,n\}};k) \geq 0$ it follows that (8) and (9) are satisfied. Note that if $f(k_1;\underline{\theta}) + b_1(\overline{\theta};\theta_n;\hat{\theta}_{-\{1,n\}}) \leq 0$, the only sufficient condition is $\beta_1\delta_1 > g_1(\overline{\theta};\lambda)$.

Consider now agent n. If agent 1 has reported $\theta_1 = \overline{\theta}$, there is nothing to check. Suppose then that he has reported $\overline{\theta}_1$. Define

$$g_n(\underline{\theta};\lambda) = u_n(f(k_n;\underline{\theta}) + \bar{b}_n + \lambda) - u_n(f(k_n;\underline{\theta}) + \bar{b}_n)$$
$$g_n(\overline{\theta};\lambda) = u_n(f(k_n;\overline{\theta}) + \bar{b}_n + \lambda) - u_n(f(k_n;\overline{\theta}) + \bar{b}_n)$$

Note that $g_n(\underline{\theta};0) = g_n(\overline{\theta};0) = 0$ and both function are strictly increasing in λ with $g_n(\underline{\theta};\lambda) > g_n(\overline{\theta};\lambda)$ if $\lambda > 0$ since u_n is strictly concave. As before, it is easy to check that incentive compatibility is satisfied if and only if

$$g_n(\underline{\theta};\lambda) \geq \beta_n\delta_n \geq g_n(\overline{\theta};\lambda) \tag{10}$$

Consider $(\lambda;\delta_n) \gg 0$ such that $g_n(\underline{\theta};\lambda) = \beta_n\delta_n$. Since then $\beta_n\delta_n > g_n(\overline{\theta};\lambda)$, incentive compatibility is then satisfied.

Step 1.4. First note that for all $i \in I \setminus \{1;n\}$, $e_i(\hat{b};\hat{k}^0;\hat{w}) = e_i(b;k^0;w)$ by construction. Also,

$$\begin{aligned} e_n(\hat{b};\hat{k}^0;\hat{w}) &= \sum_{\theta} \pi(\theta)\{u_n(f(k_n;\theta_n) + \hat{b}_n(\theta)) + \beta_n\hat{w}(\theta)\} \\ &\quad + \pi(\hat{\theta})[u_n(f(k_n;\underline{\theta}) + \bar{b}_n) - u_1(f(k_n;\underline{\theta}) + \bar{b}_n)] \\ &= e_n(b;k^0;w) + \pi(\hat{\theta})[g_n(\underline{\theta};\lambda) - \beta_n\delta_n] \\ &= e_n(b;k^0;w) \end{aligned}$$

Finally,

$$\begin{aligned} e_1(\hat b;\hat k^0;\hat w) &= \sum_{\theta} \pi(\theta)\{u_1(f(k_1;\theta_1)+\hat b_1(\theta)) + \beta_1 \hat w(\theta)\} \\ &\quad +\pi(\hat\theta)[u_1(f(k_1;\bar\theta)+\hat b_1(\hat\theta)) - u_1(f(k_1;\bar\theta)+b_1(\hat\theta))] \\ &= e_1(b;k^0;w) + \pi(\hat\theta)[\beta_1\delta_1 - g_1(\bar\theta;\lambda)] \\ &> e_1(b;k^0;w) \end{aligned}$$

Therefore, we can choose $(\lambda;\delta_1;\delta_n) \gg 0$ such that $g_n(\underline{\theta};\lambda) = \beta_n\delta_n$ and $g_1(\underline{\theta};\lambda) = \beta_1\delta_1$ such the requirements in Step 1.1 and 1.2 are satisfied. For instance, put $\lambda = 1/k$ and define (δ_1^k, δ_n^k) by letting $g_n(\underline{\theta};1/k) = \beta_n\delta_n^k > 0$ and $g_1(\underline{\theta};1/k) = \beta_1\delta_1^k > 0$; note that when $k \to \infty$; $(\delta_1^k, \delta_n^k) \to (0;0)$.

Case 2: $\bar w_n = 0$ and $w_i(\underline{\theta}_i;\bar\theta_n;\underline{\theta}_{-\{i;n\}}) > 0$ for some i ($\underline{\theta}_{-\{i;n\}}$ means that $\theta_m = \underline{\theta}$ for all $m \in I/\{i;n\}$).

First note that it follows for Lemma 6 that $f(k_n;\bar\theta) + \bar b_n > 0$. Suppose without loss of generality that $i = 1$ (we will increase agent n's utility and then use the fact that U_1^* is strictly decreasing in u_{-1}). Define $\hat\theta = (\underline{\theta}_1;\bar\theta_n;\underline{\theta}_{-\{1;n\}})$ and consider the following alternative recursive allocation:

$$\begin{aligned} \hat b_n(\hat\theta) &= \bar b_n - \lambda \text{ and } \hat b_1(\hat\theta) = b_1(\hat\theta) + \lambda \\ \hat w_n(\hat\theta) &= \bar w_n + \delta_n \text{ and } \hat w_1(\hat\theta) = w_1(\hat\theta) - \delta_1 \end{aligned}$$

If $\theta = (\underline{\theta}_1;\bar\theta_n;\underline{\theta}_{-\{1;n\}})$, then define

$$\begin{aligned} \hat b_n(\theta) &= \bar b_n \text{ and } \hat b_1(\theta) = b_1(\theta) \\ \hat w_n(\theta) &= \bar w_n \text{ and } \hat w_1(\theta) = w_1(\theta) \end{aligned}$$

Finally, de…ne for all $i \in I \setminus \{1, n\}$ and for all θ : $\hat{b}_i(\theta) = b_i(\theta)$ and $\hat{w}_i(\theta) = w_i(\theta)$.

Step 2.1. We need to look for $(\delta_1, \delta_n) \gg 0$ such that $(w_1(\hat{\theta}) - \delta_1, \overline{w}_n + \delta_n, \{w_i(\hat{\theta})\}_{i \in I \setminus \{1,n\}}) \in \Psi(k^0(\hat{\theta}))$. Note that $0 < w_1(\hat{\theta}) \leq U_1^*(\overline{w}_n, \{w_i(\hat{\theta})\}_{i \in I \setminus \{1,n\}}, k^0(\hat{\theta}))$ and then if $\delta_n > 0$ is small enough

$$0 \leq U_1^*(\overline{w}_n + \delta_n, \{w_i(\hat{\theta})\}_{i \in I \setminus \{1,n\}}, k^0(\hat{\theta}))$$

Hence, we can choose $\delta_1 > 0$ such that

$$0 \leq w_1(\hat{\theta}) - \delta_1 = U_1^*(\{\overline{w}_n + \delta_n, w_i(\hat{\theta})\}_{i \in I \setminus \{1,n\}}, k^0(\hat{\theta}))$$

In this case then $(w_1(\hat{\theta}) - \delta_1, \overline{w}_n + \delta_n, \{w_i(\hat{\theta})\}_{i \in I \setminus \{1,n\}}) \in \Psi(k^0(\hat{\theta}))$.

Step 2.2. Incentive Compatibility. As in our previous discussion, we need to check only for agent 1 and n whenever $\underline{\theta}_{-\{1,n\}}$ has been reported. Consider …rst agent 1; if agent n has reported something di¤erent from $\overline{\theta}_n$, there is nothing to check. Suppose then that agent has reported $\overline{\theta}_n$; de…ne

$$\begin{aligned} g_1(\underline{\theta}, \lambda) &= u_1(f(k_1, \underline{\theta}) + b_1(\underline{\theta}, \overline{\theta}_n, \hat{\theta}_{-\{1,n\}}) + \lambda) - u_1(f(k_1, \underline{\theta}) + b_1(\underline{\theta}, \overline{\theta}_n, \hat{\theta}_{-\{1,n\}})) \\ g_1(\overline{\theta}, \lambda) &= u_1(f(k_1, \overline{\theta}) + b_1(\underline{\theta}, \overline{\theta}_n, \hat{\theta}_{-\{1,n\}}) + \lambda) - u_1(f(k_1, \overline{\theta}) + b_1(\underline{\theta}, \overline{\theta}_n, \hat{\theta}_{-\{1,n\}})) \end{aligned}$$

Note that $g_1(\underline{\theta}, 0) = g_1(\overline{\theta}, 0) = 0$, both functions are strictly increasing in $\lambda > 0$ and $g_1(\underline{\theta}, \lambda) > g_1(\overline{\theta}, \lambda)$ whenever $\lambda > 0$.

Note that

$$\begin{aligned} & u_1(f(k_1, \underline{\theta}) + b_1(\underline{\theta}, \overline{\theta}_n, \underline{\theta}_{-\{1,n\}}) + \lambda) + \beta_1[w_1(\underline{\theta}, \overline{\theta}_n, \underline{\theta}_{-\{1,n\}}) - \delta_1] \\ \geq \; & u_1(f(k_1, \underline{\theta}) + b_1(\overline{\theta}, \overline{\theta}_n, \underline{\theta}_{-\{1,n\}}) + \beta_1 w_1(\overline{\theta}, \overline{\theta}_n, \underline{\theta}_{-\{1,n\}}) \end{aligned}$$

is satis…ed if and only if

$$l^1_{\underline{\theta}, \overline{\theta}}(\overline{\theta}_n, \underline{\theta}_{-\{1,n\}}, k) + g_1(\underline{\theta}, \lambda) \geq \beta_1 \delta_1 \qquad (11)$$

Also,

$$u_1(f(k_1;\bar{\theta}) + b_1(\bar{\theta};\bar{\theta}_n;\theta_{-\{1;n\}}) + \beta_1 w_1(\bar{\theta};\bar{\theta}_n;\theta_{-\{1;n\}})$$

$$\geq \quad u_1(f(k_1;\bar{\theta}) + b_1(\underline{\theta};\bar{\theta}_n;\theta_{-\{1;n\}}) + \lambda) + \beta_1[w_1(\underline{\theta};\bar{\theta}_n;\theta_{-\{1;n\}}) - \delta_1]$$

holds if and only if

$$l^1_{\bar{\theta};\underline{\theta}}(\bar{\theta}_n;\theta_{-\{1;n\}};k) + \beta_1\delta_1 \geq g_1(\bar{\theta};\lambda) \tag{12}$$

Define $(\delta_1;\lambda) \gg 0$ such that $g_1(\underline{\theta};\lambda) = \beta_1\delta_1$ and note that then (12) and (13) are satisfied.

Consider agent n now and suppose that agent 1 has reported $\underline{\theta}$ (otherwise there is nothing to check). As before, if $f(k_n;\underline{\theta}) + \bar{b}_n \leq 0$ and $\lambda > 0$; there is only one incentive compatibility to check. Consider the more general case where $f(k_n;\underline{\theta}) + \bar{b}_n > 0$: Define the corresponding

$$g_n(\underline{\theta};\lambda) \;=\; u_n(f(k_n;\underline{\theta}) + \bar{b}_n) - u_n(f(k_n;\underline{\theta}) + \bar{b}_n - \lambda)$$

$$g_n(\bar{\theta};\lambda) \;=\; u_n(f(k_n;\bar{\theta}) + \bar{b}_n) - u_n(f(k_n;\bar{\theta}) + \bar{b}_n - \lambda)$$

Note that

$$u_n(f(k_n;\underline{\theta}) + \bar{b}_n) \geq u_1(f(k_n;\underline{\theta}) + \bar{b}_n - \lambda) + \beta_n\delta_n$$

and

$$u_n(f(k_n;\bar{\theta}) + \bar{b}_n - \lambda) + \beta_n\delta_n \geq u_1(f(k_n;\bar{\theta}) + \bar{b}_n)$$

hold if and only if

$$g_n(\underline{\theta};\lambda) \geq \beta_n\delta_n \tag{13}$$

and

$$\beta_n\delta_n \geq g_n(\bar{\theta};\lambda) \tag{14}$$

are satisfied. Define $(\delta_n;\lambda) \gg 0$ such that $g_n(\underline{\theta};\lambda) = \beta_n\delta_n$ and note that then (14) and (15) are satisfied.

Step 2.3. For all $i \in I=\{1;n\}$; $e_i(\hat{b};\hat{k}';\hat{w}) = e_i(b;k';w)$ by construction. Also,

$$\begin{aligned} e_n(\hat{b};\hat{k}';\hat{w}) &= \sum_{\theta} \pi(\theta)\{u_n(f(k_n;\theta_n)+\hat{b}_n(\theta)) + \beta_n\hat{w}(\theta)\} \\ &\quad + \pi(\hat{\theta})[u_n(f(k_n;\underline{\theta})+\bar{b}_n) - u_1(f(k_n;\underline{\theta})+\bar{b}_n)] \\ &= e_n(b;k';w) - \pi(\hat{\theta})[g_n(\bar{\theta};\lambda) - \beta_n\delta_n] \\ &> e_n(b;k';w) \end{aligned}$$

Finally,

$$\begin{aligned} e_1(\hat{b};\hat{k}';\hat{w}) &= \sum_{\theta} \pi(\theta)\{u_1(f(k_1;\theta_1)+\hat{b}_1(\theta)) + \beta_1\hat{w}(\theta)\} \\ &= e_1(b;k';w) + \pi(\hat{\theta})[g_1(\underline{\theta};\lambda) - \beta_1\delta_1] \\ &= e_1(b;k';w) \end{aligned}$$

Define $\hat{u}_n = e_n(\hat{b};\hat{k}';\hat{w})$ and $\hat{u}_{-1} = (\hat{u}_n;\{u_i\}_{i\in I\neq\{1;n\}})$: As before, $(\lambda;\delta_1;\delta_n) \gg 0$ can be appropriately chosen such $(\hat{b};\hat{k}';\hat{w})$ is admissible with respect to Ψ at k and promise keeping at $\hat{u}_{-1}$: But then $e_1(b;k';w) \leq U_1^*(\hat{u}_{-1};k) < U_1^*(u_{-1};k) = e_1(b;k';w)$ since U_1^* is strictly decreasing in u_{-1}: This is the desired contradiction.

Case 3. $\bar{w}_n = 0$ and $w_i(\theta_i;\bar{\theta}_n;\theta_{-\{i;n\}}) = 0$ for all $i \neq n$:

This case is similar to case 2 but simpler after observing the following facts. Define $\hat{\theta} = (\bar{\theta}_n;\theta_{-n})$ and note then that $w_i(\hat{\theta})$ for all agent $i \in I$: From Remark 2 we know that $U_1^*(0;k) > 0$ for all $k \in X$: Therefore, there exists some $\delta_n > 0$ such that $(0;\dots;\delta_n;\dots 0) \in \Psi(k)$: Proceeding as in Case 2 but just redefining $b_n(\hat{\theta})$ and $w_n(\hat{\theta})$ for agent n one can prove this part. Details are left to the reader. ∎

Proof of Proposition 4. Denote $\Delta(k;\overline{W}_n) = \{\{\theta_t\}_{t=0}^{\infty} \in \Omega : \lim_{t\to\infty} W_{nt}(\theta^t) = \overline{W}_n \in \Psi_n(k)\}$. Take any $\{\theta_t\}_{t=0}^{\infty} \in \Delta(k;\overline{W}_n)$ and consider the path of the following vector

$$\{K_t; W_t; b^*(\theta)(K_t;W_t); k'^*(\theta)(K_t;W_t); w^*(\theta)(K_t;W_t)\}_{t=0}^{\infty}$$

where $w^*(\theta)(K_t;W_t) \in \Psi(k'^*(\theta)(K_t;W_t))$ for all t and all $\theta \in \Theta^N$ and for all t

$$\begin{aligned} W_{nt} &= \sum_{\theta \in \Theta^N} \{u_n(f(k_{nt};\theta_n) + b_n^*(\theta)(K_t;W_t)) + \beta_n w_n^*(\theta)(K_t;W_t)\} \\ W_{1t} &= U_1^*(K_t;W_{-1t}) \end{aligned}$$

Note now that the considered path is a sequence is in a compact set and therefore it will have a convergent subsequence. Without loss of generality, assume that the relevant subsequence is the sequence itself. Denote the corresponding limit point by $\{\hat{K};\hat{W};\hat{b};\hat{k}';\hat{w}\}$. Note that $\hat{W}_n = \overline{W}_n$.

Step 1. $(\hat{b};\hat{k}';\hat{w})$ is admissible with respect to $\Psi(\hat{k})$ at $\hat{w}$. Moreover, it is efficient at $(\hat{k};\hat{W}_{-1})$.

To see this, note first that, by definition, $(w^*(\theta)(K_t;W_t); k'^*(\theta)(K_t;W_t)) \in \mathrm{graph}(\Psi)$ for all t and all $\theta \in \Theta^N$. Since Ψ has a compact graph, it follows that for all $\theta \in \Theta^N$ $(\hat{w}(\theta);\hat{k}'(\theta)) \in \mathrm{graph}(\Psi)$ and then $\hat{w}(\theta) \in \Psi(\hat{k}'(\theta))$ for all $\theta \in \Theta^N$.

Since weak inequalities are preserved in the limit and continuity of f, it is also true that for all $\theta \in \Theta^N$

$$\begin{aligned} \sum_{n \in I} [\hat{b}_n(\theta) + \hat{k}'(\theta)] &\leq 0 \\ f(\hat{k};\theta_n) + \hat{b}_n(\theta) &\geq 0 \end{aligned}$$

By continuity of f and u_n, for all $n \in I$

$$\lim_{t\to\infty} W_{nt} = \lim_{t\to\infty} \sum_{\theta\in\Theta^N} \{u_n(f(k_{nt},\theta_n) + b_n^*(\theta)(K_t,W_t)) + \beta_n w_n^*(\theta)(K_t,W_t)\}$$

$$\hat{W}_n = \sum_{\theta\in\Theta^N} \{u_n(f(\hat{k}_n,\theta_n) + \hat{b}_n(\theta)) + \beta_n \hat{w}_n(\theta)\}$$

Finally, note that since U_1^* is assumed to be continuous, it follows that

$$\lim_{t\to\infty} W_{1t} = U_1^*(K_t, W_{-1t}) = U_1^*(\hat{k}, \hat{W}_{-1})$$

That is, $(\hat{b}, \hat{k}', \hat{w})$ is an efficient recursive allocation at $(\hat{k}, \hat{W}_{-1})$

Step2. There exists some $\hat{\theta}_{-n}$ such either

$$\text{(a)} \qquad \lim_{t\to\infty} w_n^*(\underline{\theta}, \hat{\theta}_{-n})(K_t, W_t) = \overline{W}_n$$

or

$$\text{(b)} \qquad \lim_{t\to\infty} w_n^*(\overline{\theta}, \hat{\theta}_{-n})(K_t, W_t) = \overline{W}_n$$

To see this, assume that it is not true. Then, for all (θ_n, θ_{-n}) note that given Lemma 4

$$\overline{W}_n = \lim_{t\to\infty} w_n^*(\underline{\theta}, \theta_{-n})(K_t, W_t) \geq \lim_{t\to\infty} w_n^*(\theta_n, \theta_{-n})(K_t, W_t)$$
$$\geq \lim_{t\to\infty} w_n^*(\overline{\theta}, \hat{\theta}_{-n})(K_t, W_t) = \overline{W}_n$$

Now observe that for all $\theta_n, \hat{\theta}_n$, and θ_{-n}, incentive compatibility implies

$$\beta_n[w_n^*(\hat{\theta}_n, \theta_{-n})(K_t, W_t) - w_n^*(\theta_n, \theta_{-n})(K_t, W_t)]$$
$$\geq u_n(f(k_{nt},\theta_n) + b_n^*(\theta_n, \theta_{-n})(K_t, W_t)) - u_n(f(k_{nt},\theta_n) + b_n^*(\hat{\theta}_n, \theta_{-n})(K_t, W_t))$$
$$\geq \beta_n[w_n^*(\hat{\theta}_n, \theta_{-n})(K_t, W_t) - w_n^*(\theta_n, \theta_{-n})(K_t, W_t)]$$

Taking limits it follows that

$$\lim_{t\to\infty} [u_n(f(k_{nt},\theta_n) + b_n^*(\theta_n, \theta_{-n})(K_t, W_t)) - u_n(f(k_{nt},\theta_n) + b_n^*(\hat{\theta}_n, \theta_{-n})(K_t, W_t))] = 0$$

Since u_n is assumed continuous and strictly increasing, f is continuous and

$$(k_{nt}; b^{¤}_n(\mu_n; \mu_{¡n})(K_t; W_t)) \ ! \ (\overline{k}_n; \overline{b}_n(\mu_n; \mu_{¡n}))$$

it follows that $\overline{b}_n(\mu_n; \mu_{¡n}) = \overline{b}_n$ for all $(\mu_n; \mu_{¡n})$: But this means that there is a the result in Proposition 3.

Step 3. Suppose that (a) holds and consider the sequence of q^0s such that

$$W_{nt_q+1} = w^{¤}_n(\underline{\mu}; \mu_{¡n})(K_{t_q}; W_{t_q})$$

Since (a) holds, this equality can hold only for a ...nite number of q^0s: Therefore,

$$\Delta(k; \overline{W}_n) \ ½ \ ff\mu_t g^1_{t=0} g \ 2 \ \Omega : \mu_t = (\underline{\mu}; \mu_{¡n}) \text{ finitely ofteng}$$

but $P ff\mu_t g^1_{t=0} g \ 2 \ \Omega : \mu_t = (\underline{\mu}; \mu_{¡n}) \text{ finitely ofteng} = 0$: ■

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