Universidad de
SanAndrés

Seminario del Departamento de Economía

"Multidimensional Mechanism Design: Optimal Pricing in a Multiple-Good Monopoly"

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Martes 26 de junio de 2001
Aula del CEDI, de 11 a 12:30hs.

Multidimensional Mechanism Design: Optimal Pricing in a Multiple-Good Monopoly

Preliminary and Incomplete. Draft prepared for presentation at the Universidad de San Andres.

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Draft: June 22, 2001

Abstract

To be added.

Keywords: auctions, multi-dimmensional, incentive compatibility, adverse selection.

1 The Model

A seller with N different objects faces a single buyer. The seller does not observe the buyer's valuation for each object; valuations are the buyer's private information.

The buyer's preferences over consumption and money transfers are given by

$$U(x, q, t) = x \cdot q - t,$$

where x is the vector of buyer's valuations, q is the quantity consumed of each good, and t is the monetary transfer made to the seller. Since the buyer has demand for at most one unit of each good, the vector q is an element of $\{0, 1\}^L$; for ease of notation we assume that x is an element of I^N where $I = [0, 1]$, and t is in $\mathbb{R}$.

The prior beliefs about the buyer's valuation x is given by a strictly-positive, density function $f(x)$, which is common knowledge and represents the seller's believes about the buyer's private information.

In searching for an optimal mechanism, one may restrict attention to direct revelation mechanisms where buyers report their types truthfully. A direct revelation mechanism is a pair of functions

$$p : I^N \longrightarrow I^N$$
$$t : I^N \longrightarrow \mathbb{R},$$

where $p_i(x)$, the i^{th} component of $p(x)$, is the probability that the buyer will obtain good i when her valuation is x, and $t(x)$ is the transfer made by the buyer to the seller when valuations are x.[1] In addition, the buyer must have adequate incentives to reveal its information truthfully—incentive compatibility (IC)—and to participate in the mechanism voluntarily—individual rationality (IR). The buyer's expected payoff $u(x'|x)$ under the mechanism (p, t) when the buyer has valuation x and reports x' is

$$u(x'|x) = p(x') \cdot x - t(x').$$

For ease of notation, $u(x|x)$ is denoted $u(x)$. Then, (p, t) must satisfy

$$\begin{array}{ll} \textbf{(IC)} & \forall x, u(x) \geq u(x'|x)\ \forall x' \\ \textbf{(IR)} & \forall x, u(x) \geq 0. \end{array}$$

(As stated, the constraints hold everywhere; it suffices that they hold almost everywhere.)

We summarize in a lemma some readily available properties of IC and IR mechanisms that have been noted and used in the literature.[2]

Lemma 1 ??

[1] In order to compute expected payoffs, the functions p and t must be integrable.

[2] See Rochet (1984), Armstrong (1998), and Jehiel, Moldovanu, and Stacchetti (1998), Krishna and Maenner (2000).

1. *If (p,t) is a mechanism that satisfies IC, then the buyer's expected payoff $u(x)$ is convex, and its differential $\nabla u(x) = \left[\frac{\partial u(x)}{\partial x_i}\right]$ belongs to I^N for most $x \in I^N$. Indeed $\nabla u(x) = p(x)$ almost everywhere.*[3]

2. *If $\varphi(x)$ is a convex function, and its differential $\nabla \varphi(x) = \left[\frac{\partial \varphi(x)}{\partial x_i}\right] \in I^N$ for most $x \in I^N$, then there exist a mechanism (p,t) satisfying IC. The mechanism is defined by $p(x) = \nabla \varphi(x)$ almost everywhere, and $t(x) = p(x) \cdot x - \varphi(x)$. Under these definitions, $u(x) = \varphi(x)$.*

Intuitively, a mechanism is IC if and only if the corresponding buyer's payoffs are convex, with partial derivatives between zero and one.

The preceding properties completely characterize IC mechanisms in terms of the buyer's expected-payoff function $u(x)$. Individual rationality requires that u be non-negative. Since the objective is to find an optimal policy for the buyer, and since the buyer's expected payoff is non-decreasing, there is no loss of generality in restricting attention to payoff function where $u(0) = 0$. **This requires explanation.***

The *set of IC, IR mechanisms* is

$$W = \left\{ u \in C^1(I^N) \mid u(x) \text{ is convex, } \nabla u(x) \in I^N \text{ } a.e., \text{ and } u(0) = 0 \right\}.$$

For later reference, we summarize in Lemma 2 below several simple properties of IC, IR mechanisms.

Lemma 2 *If u belongs to W, then*

1. *$u(x)$ is non-negative for all x.*

2. *u is non-decreasing: $x' \geq x \implies u(x') \geq u(x)$.*

3. *u is continuous.*

4. *u is a.e. differentiable.*

5. *u is monotone: $(\nabla u(x') - \nabla u(x)) \cdot (x' - x) \geq 0$, for all x', x.*

Given any IC, IR mechanism $u(\cdot)$, a buyer with type x receives a payoff $u(x) = \nabla u(x) \cdot x - t(x)$. The seller's expected revenue when using the mechanism $u(\cdot)$ is

$$E[t(x)] = E\left[\nabla u(x) \cdot x - u(x)\right].$$

The seller's problem is to select a mechanism $u \in W$ to maximize expected revenue,

$$\max_{u \in W} E[\nabla u(x) \cdot x - u(x)]. \tag{1}$$

[3]Since $u(x)$ is convex, it is a continuous function, and almost everywhere differentiable.

2 Optimal Mechanisms

An optimal mechanism is a solution to the maximization problem (1) described in the previous section. We consider in turn the structure of the feasible set W and the nature of the objective function.

Any function obtained as a convex combination of elements of W is convex, non-negative, and satisfies the bounds on partial derivatives (its gradient takes values in I^N). Thus, W is itself a convex set. It is also simple to verify that W is compact with respect to the sup-norm (Lemma 4 in the Appendix). Therefore, W has extreme points (Krein-Milman Theorem).

The objective function of the seller's problem is linear on the mechanism $u(\cdot)$. Thus, a solution can always be found on an extreme point of W (Bauer Maximum Principle, see for instance Aliprantis and Border (1999), page 230).

Information about the extreme points of the set W of IC, IR mechanisms can be very useful in identifying properties of the optimal mechanism. A *zero-one mechanism* $u(\cdot) \in W$ is a mechanism in which each object is always either assigned for certain or not at all; i.e. the probability of trade $\nabla u(x) \in \{0, 1\}$ for all $x \in I^N$. Zero-one mechanisms do not "randomize" the assignment of objects.

The following simple fact illustrates the potential usefulness of the previous discussion.

Theorem 1 *If the seller has a single good, i.e., $N = 1$, an IC, IR mechanism is an extreme point if and only if it is a zero-one mechanism.*

Proof Zero-one mechanisms are clearly extreme points: for any continuous g with $\nabla g(x) \neq 0$, either $u + g$ or $u - g$ is not in W. Thus, ∇g must be a.e. zero. To establish the converse select any $u \in W$ that is not a zero-one mechanism. Then, there is a set of positive measure $B \subset [0, 1]$ such that $\epsilon < \nabla u(x) < 1 - \epsilon$. Let

$$\nabla g(x) = \begin{cases} 1 - \nabla u(x) & \text{if } \nabla u(x) > 0.5 \\ \nabla u(x) & \text{if } \nabla u(x) \leq 0.5 \end{cases}$$

Let $g(x) = \int_0^x \nabla g(z)dz$; then $g(x)$ is a continuous function. We now verify that both $u + g$ and $u - g$ are in W. First, the gradient of $u + g$ is in $[0, 1]$:

$$\nabla(u(x) + g(x)) = \begin{cases} 1 & \text{if } \nabla u(x) > 0.5 \\ 2\nabla u(x) & \text{if } \nabla u(x) \leq 0.5 \end{cases}$$

Second, $\nabla(u(x) + g(x))$ is increasing in x, $u + g$ is convex. Third, $g(0) = 0$ by construction. Thus $u + g$ is in W.

A similar argument applies to $u - g$.

Q.E.D.

The characterization of the set of extreme points of W readily provide alternative proofs of various well known results in one-dimensional environments. For instance, optimal bargaining mechanism for an uninformed seller is a take-it-or-leave-it offer (Samuelson 1981), randomization (i.e. no "huggling") need not be part of an optimal selling strategy (Maskin and Rile 1984), etc. These well known results do not extend to higher dimensions.

In one-dimensional environments, the set of extreme points of W is relatively small, the set of zero-one mechanisms. In higher dimensions (i.e., $N \geq 2$), the set of extreme points of W is considerably richer; it may involve randomization.

2.1 Examples

Example 1: An extreme point that involves randomization.

Consider the set W of IC, IR mechanisms in two dimensions. Let $u \in W$ be defined by

$$u(x) = \max\{0,\ (0.5x_1 - 0.2),\ (x_1 + x_2 - 1)\}.$$

The graph of u is depicted in Figure 1.

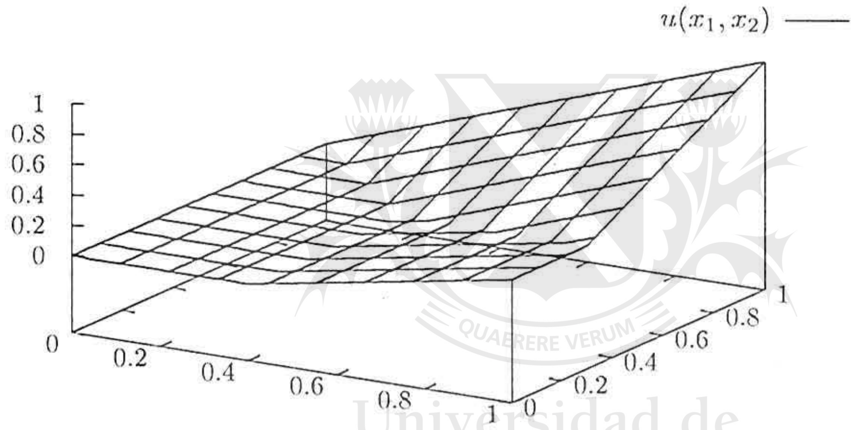

Figure 1: $u(x) = \max\{0,\ (0.5x_1 - 0.2),\ (x_1 + x_2 - 1)\}$

The mechanism u is composed of three linear pieces. The effective domain of each linear piece is depicted in Figure 2. The symbol $A_{i,j}$ represents the set of types x where $\nabla u(x) = (i, j)$.

A general argument, presented later on this section, demonstrates that the mechanism u is an extreme point of W. For the moment, the following intuition may suffice the curious reader. If u is not an extreme point, then there is a function $g \neq 0$ such that $u \pm g \in W$. Since $u \pm g$ is is continuous, and a.e. differentiable (Lemma 2(4) and (5)); g must be continous, and a.e. differentiable. Notice that for $x \in A_{0,0} \cup A_{1,1}$, ∇g must be identically zero; otherwise either $\nabla(u + g)$ or $\nabla(u - g)$ is not in I^N. It follows by continuity that $g(x) = 0$ for all $x \in A_{0,0} \cup A_{1,1}$. If $g(x) > 0$ for some $x \in A_{.5,0}$, then since $u + g$ is non-decreasing (Lemma 2(2)) , $g(x) > 0$ for any $x \in A_{1,1,} \cap A_{.5,0}$. This is a contradiction.

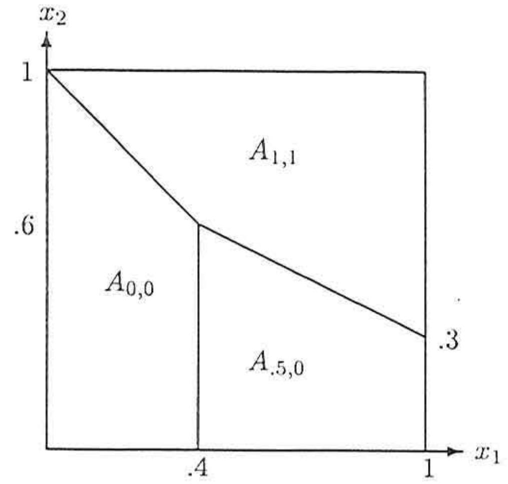

Figure 2: Defining Partition

Example 2: An extreme point that is not piece-wise linear.

Suppose there are two objects. Let $u \in W$ be defined by

$$u(x) = \max\{0,\ (0.25x_1^2 + x_2 - 0.5)\}.$$

The graph of u is depicted in Figure 3.

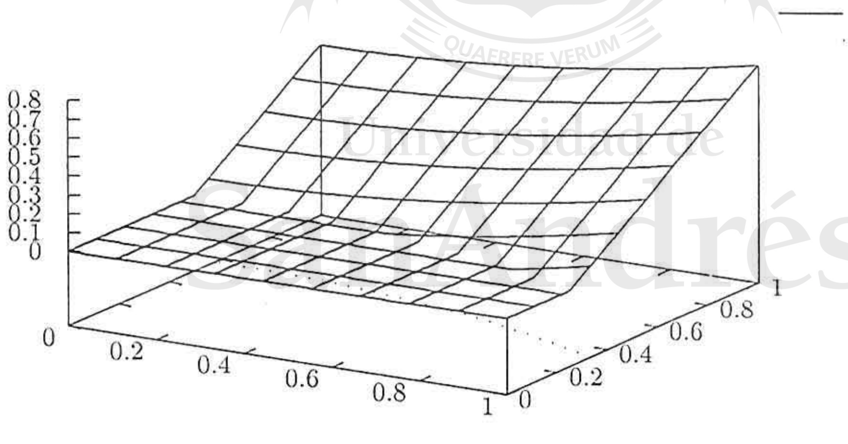

Figure 3: $u(x) = \max\{0,\ (0.25x_1^2 + x_2 - 0.5)\}$

Suppose u is not an extreme point. Then there is a function $g(x)$ such that $u \pm g \in W$. Using a similar argument to that employed in Example 1, it follows that $\nabla g(x) = 0$ for $x \in A_{0,0}$, and that $\nabla_2 g(x) = 0$ for all x. By continuity, $g(x) = 0$ for all x in the boundary of $A_{0,0}$. ** needs some

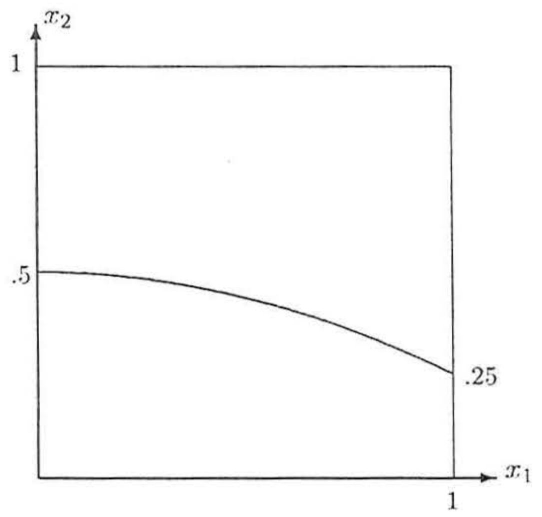

Figure 4: $x_2 = 0.5 - 0.25x_1^2$

work *** If g is different from zero, however, it must have $\nabla_1 g(x) \neq 0$ for some x. In $I^2 \setminus A_{0,0}$, $|\nabla g(x)| \leq \max\{.5x_1, (1 - .5x_1)\}$.

2.2 Higher Dimensions

The set C is *a convex cone with vertex* $\bar{\eta}$ if it is convex and for all $\eta \in C$, $\bar{\eta} + k\eta \in C$ for all $k \in I\!R_+$.

Theorem 2 *Let C be a convex cone with vertex $\bar{\eta}$ in a locally convex, topological vector space X. Suppose that*

$$\eta \in C, \eta \neq 0 \implies -\eta \notin \bar{C}. \tag{2}$$

Then there exists a continuous linear functional $f \neq 0$ such that $\langle f, \bar{\eta}\rangle > \langle f, \eta\rangle$, $\forall \eta \in \bar{C}, \eta \neq 0$

Proof By translating C if necessary, we may assume without loss of generality that $\bar{\eta} = 0$. We must therefore show that there is a continuous linear functional f such that $0 \geq \langle f, \eta\rangle$ for all $\eta \in \bar{C}$, $\eta \neq 0$.

If C is either empty or a singleton, the theorem is trivial. If C has more than one element, C is not dense by hypothesis. It follows that $\bar{C}$ is the intersection of all topologically closed half spaces containing it (see for instance, Aliprantis and Border (1999), Corollary 5.62, page 194).

A half space is any set of the form $[f \leq r] = \{\eta \in X : \langle f, \eta\rangle \leq r\}$ where f is any non-zero continuous linear functional and $r \in I\!R$. Note that if $[f \leq r]$ contains C, then $r \geq 0$, and $[f \leq 0]$ also contains C.

Thus, $C = \bigcap_{\alpha \in A}[f_\alpha \leq 0]$. Define

$$\tilde{C} = \bar{\text{co}}\{C \setminus \bigcap_{\alpha \in A}[f_\alpha \leq -\epsilon]\}.$$

We now show that $0 \notin \tilde{C}$. Suppose to the contrary that 0 is an element of $\tilde{C}$. Then there is a sequence η^n such that $\eta^n \longrightarrow 0$, and $\eta^n = \sum_{k \in K^n} \beta_k^n \eta_k^n$, where $\beta_k^n \geq 0$ for all n and k, K^n is a finite set of indecis, $\sum_{k \in K^n} \beta_k^n = 1$, and $\eta_k^n \in (C \setminus \bigcap_{\alpha \in A} [f_\alpha \leq -\epsilon])$ for all n and k.

For any $\alpha \in A$,

$$\langle f_\alpha, \eta^n \rangle = \langle f_\alpha, \sum_{k \in K^n} \beta_k^n \eta_k^n \rangle = \sum_{k \in K^n} \beta_k^n \langle f_\alpha, \eta_k^n \rangle < \sum_{k \in K^n} \beta_k^n (-\epsilon) \leq -\epsilon.$$

Since f_α is continuous, however, $\langle f_\alpha, \eta^n \rangle$ must converge to zero as n approaches infinity, a contradiction.

Since $0 \notin \tilde{C}$, there is a continuous linear functional f such that

$$0 > \sup\{\langle f, \eta \rangle : \eta \in \tilde{C}\}$$

(for instance, Aliprantis and Border (1999), Corollary 5.59, page 194).

We complete the proof by showing that $0 \geq \langle f, \eta \rangle \forall \eta \in C, \eta \neq 0$. The argument shows that if the inequality is not satisfied for some $\eta' \in C$, then there is a $k\eta'$ in $\tilde{C}$ that also violates it.

Suppose there is a non-zero $\eta' \in C$ with $\langle f, \eta' \rangle \geq 0$. For any $k \in I\!R_+$, the element $k\eta'$ belongs to C and $-k\eta'$ is not in $\bar{C}$. Then, there is a continuous linear functional g such that $\langle g, -\eta' \rangle > \sup\{\langle g, \eta \rangle : \eta \in \bar{C}\} = 0$. (We used the fact that $0 \in \bar{C}$.) Note that the half space $[g \leq \langle g, -\eta' \rangle]$ contains $\bar{C}$, and $[g \leq 0]$ must also contain $\bar{C}$. It follows that $\langle g, \eta \rangle < 0$ and that for a sufficiently large $k > 0$, $\langle g, k\eta' \rangle < \epsilon$. In turn this implies that $k\eta' \in \tilde{C}$. Since $\langle f, k\eta' \rangle = k\langle f, \eta' \rangle \geq 0$, we have a contradiction. *Q.E.D.*

We first show that it is possible to restrict attention to the set of piece-wise linear mechanisms in W.

A mechanism u is *piecewise linear* if there exist a partition of I^N such that u is linear in each element of the partition. Thus, u is piecewise linear if and only if u is the pointwise maximum of finitely many linear functions.

Theorem 3 *The set of piecewise linear mechanisms in W is dense in W.*

Given any IC and IR mechanism $u \in W$, the seller's revenue from transacting with a buyer of type x is $v_u = \nabla u(x) \cdot x - u(x)$.

A mechanism $u \in W$ is *dominated* if there is an alternative mechanism $u' \in W$ such that $v_{u'}(x) \geq v_u(x)$ for all $x \in I^N$ with strict inequality in set of positive Lebesgue measure. Dominated mechanisms are trivially suboptimal. Even extreme points of W can be dominated. To see this consider, the mechanism in which no buyer ever gets an object (i.e., $u(x) = 0$) and the mechanism in which buyers always get the object (i.e., $\bar{u}(x) = 1$). These two mechanisms are extreme points of W; the probabiltiy of trade $\nabla u(x)$ equals zero and one respectively. Both mechanisms, however, yield zero revenue to the seller independent of the buyer's type. They are easily dominated by $u'(x) = \max\left\{0, (1 \cdot x - \frac{N}{2})\right\}$.

In the following discussion we show that any undominated, extreme point of W is an optimal mechanism for an appropriate distribution of types.

Theorem 4 *Let $\bar{u}$ be an undominated, extreme point of W. Then there is a density f such that $\bar{u}$ is the optimal selling mechanism.*

Proof Let

$$V = \{v_u \ : \ v_u(x) = \nabla u(x) \cdot x - u(x) \ \text{ for some } u \in W\}.$$

It is simple to establish (by verifying definitions) that V is convex and that for any extreme point $\bar{u} \in W$, $v_{\bar{u}}$ is an extreme point of V. (For ease of notation, we write v and $\bar{v}$ instead of v_u and $v_{\bar{u}}$.)

Let C be the pointed convex cone with vertex $\bar{\mu}$ generated by V:

$$C = \{\bar{v} + k(v - \bar{v}) : \ k \in I\!R_+, \ v \in V\}.$$

It follows as an application of the Hahn-Banach Extension Teorem (see for instance, Lemma 5.72 (due to Klee) in Aliprantis and Border (1999), page 201) that *in a locally convex space a convex cone is supported (by a continuous linear functional) at its vertex if and only if the cone is not dense.*

We now show that the cone C is not dense as a subset of L_∞. Suppose to the contrary that C is dense and let $\tilde{v}(x) = 3N$. Then, there is a sequence $\{\bar{v} + k^n(v^n - \bar{v})\} \in C$ such that

$$\forall \epsilon > 0, \ \exists \bar{n} \ \text{ such that } \ n > \bar{n} \text{ implies } \|\tilde{v} - \bar{v} + k^n(v^n - \bar{v})\|_\infty < \epsilon.$$

For each n, since $\bar{v}$ is not dominated by v^n, there is a set $G \subset I^N$ with $v^n(G) < \bar{v}(G)$. Then, note that

$$\begin{aligned}
\|\tilde{v} - \bar{v} - k^n(v^n - \bar{v})\|_\infty &\geq \|[\tilde{v} - \bar{v} - k^n(v^n - \bar{v})]1_G\|_\infty \\
&\geq \|[\tilde{v} - \bar{v}]1_G\|_\infty \\
&\geq (3N - N) > 0,
\end{aligned}$$

where the inequality follows because $G \subset I^N$, and the second one because both terms are positive for $x \in G$: $\tilde{v}(x) - \bar{v}(x) \geq 3N - N > 0$ $(\forall x \in G)$ and $-k^n(v^n(x) - \bar{v}(x)) > 0$ $(\forall x \in G)$.

Let P denote the positive cone of $L_\infty(I^N)$. Note that the same argument above shows that the cone $C - P$ is not dense.

We conclude that

$$\exists g \neq 0, \ g \in L_1(I^N), \ \text{ such that } \ \langle \bar{v}, g \rangle \geq \langle v, g \rangle \ \forall v \in V.$$

Note that $g(x) \geq 0$ almost everywhere. This is so because the negative orthant $-P$ is separated by g. **This needs explanation ***. Thus, defining

$$f(x) = \frac{g(x)}{\int g(x)\, d\lambda},$$

we obtained the desired density f supporting $\bar{v}$; $\bar{u}$ is an optimal mechanism with respect to f. *Q.E.D.*

Given a mechanism $u \in W$ and a set $B \subset I^N$ of type realizations, the average contribution to seller's revenue of transacting with agents in B is

$$\mu_u(B) = \int_B [\nabla u(x) \cdot x - u(x)]dx.$$

The expression above $\mu_u(\cdot)$ defines a finite measure on the Borel subsets of I^N. Let

$$V^m = \{\mu_u |\ u \in W\}.$$

Thus the set V is a subset of $ca(I^N)$, the vector space of bounded, signed measures on I^N.

It is simple to establish (by verifying definitions) that V^m is convex and that for any extreme point $\bar{u} \in W$, there is a corresponding extreme point $\bar{\mu}_{\bar{u}} \in V$. (For ease of notation, we sometimes use μ and $\bar{\mu}$ instead of μ_u and $\bar{\mu}_{\bar{u}}$.)

Theorem 5 *Let $\bar{\mu} \in V^m$ be generated by $u \in W$. Suppose there is $B \subset I^N$ such that $\bar{\mu}(B) \geq \mu(B)$ for all $\mu \in V^m$. Then there is a continuous density function f such that $\bar{u}$ is the optimal selling mechanism.*

Proof

Let C be the pointed convex cone with vertex $\bar{\mu}$ generated by V^m, i.e.,

$$C = \{\bar{\mu} + k(\mu - \bar{\mu}) : \ k \in I\!R_+,\ \mu \in V\}.$$

It follows as an application of the Hahn-Banach Extension Teorem (see for instance, Lemma 5.72 (due to Klee) in Aliprantis and Border (1999), page 201) that *in a locally convex space a convex cone is supported (by a continuous linear functional) at its vertex if and only if the cone is not dense.*

We now show that the cone C is not dense with respect to the weak* topology. Let λ be the Lebesgue measure in I^N. We will show that no sequence in C can approximate $3N\lambda$. Suppose to the contrary that there is a sequence $\bar{\mu} + k^n(\mu^n - \bar{\mu})$, $= 1, 2, \ldots$, that converges (weak*) to $3N\lambda$. Then, as $n \longrightarrow \infty$, we observe that

$$k^n(\mu^n - \bar{\mu}) \xrightarrow{w^*} (3N\lambda - \bar{\mu}).$$

Note, however, that the sequence μ^n has a norm converging subsequence (Lemma 6 in the Appendix). Thus, $(\mu^n - \bar{\mu})$ converges to $(\mu - \bar{\mu})$ in norm. Therefore,

$$k^n(\mu^n - \bar{\mu}) \xrightarrow{w^*} (3N\lambda - \bar{\mu}).$$

Then, for any open rectangle $O \in I^N$,

$$\bar{u}(O) + k^n(\mu^n(O) - \bar{\mu}(O)) \longrightarrow 3N\lambda(O).$$

(Note that the boundaries of open rectangles have λ-measure zero.)

Note first that for any $\mu \in V$ and any subset $O \subset I^N$, $\mu(O) \leq N\lambda(O)$. (This follows because $\mu(O) \int_O [\nabla u(x) - u(x)]\, d\lambda \leq \int_O (1 \cdot x)\, d\lambda \leq \int_O N\, d\lambda$.) Note also that $\lambda(O) > 0$ for any open subset O of I^N. This implies that

$$k^n(\mu^n(O) - \bar{\mu}(O)) \longrightarrow [(N+1)\lambda(O) - \bar{\mu}(O)] > 0.$$

Thus for any open set $O \subset I^N$, there is $\bar{n}$ such that $n > \bar{n}$ implies

$$k^n(\mu^n(O) - \bar{\mu}(O)) > 0.$$

Setting $O = B_0$

Note also that since V is compact, we may assume that, in a subsequence if necessary, $\mu^n \Longrightarrow \mu$. Suppose $\mu \neq \bar{\mu}$. Then,

Let P denote the positive cone of $ca(I^N)$, i.e., the set of positive bounded measures.

It is simple to check that $C - P$ is also a pointed cone with vertex $\bar{\mu}$.

A separating hyperplane argument yields the desired result: the cone $C - P$ is supported at its vertex if and only if the cone $C - P$ is not dense (Lemma 5.72 (due to Klee), Aliprantis and Border (1999), page 201).

We now verify that the cone is not dense. We'll show that the positive measure 3λ (where λ is Lebesgue measure) is not approximated by elements of $C - P$.

———————————————— Outline

1. The set of piece-wise linear mechanisms is dense. Thus, to some extent, one can restrict attention to piece-wise linear mechanisms.
2. The set convex hull of the set of zero-one mechanisms is not dense in W.
3. Undominated mechanisms can be solutions to optimization problems.
4. Mechanisms that are optimal in any subset can be supported by continuous densities.
5. ————————————————

3 Appendix

Lemma 3 *Let $u^n \in W$ for $n = 1, 2, \ldots$. If the sequence u^n converges pointwise to a function u, $u : I^N \longrightarrow I\!R$, then u^n converges uniformly to u, and $u \in W$.*

Proof To be added.

Lemma 4 *The set W of individually rational (IR) and incentive compatible (IC) mechanisms is compact with respect to the sup norm.*

Proof The family of functions W is equicontinuous and uniformly bounded. The Arzela-Ascoli Theorem (see for instance, Royden (1968), page 179) implies the desired result.

Q.E.D.

Lemma 5 *Let $u^n, n = 1, 2, \ldots$ be a sequence in W that converges uniformly to $u \in W$. Then the sequence of gradients ∇u^n converges pointwise λ-a.e. to ∇u*

Proof For $n = 1, 2, \ldots$, let D^n be the set of $x \in I^N$ where $u^n(x)$ is differentiable, and let D' be similarly defined for u. The sets just defined are dense in I^N and have λ measure one (Rockafellar (1970), Theorem 25.5, page 246); furthermore The set $D = (\bigcap_{n\geq 1} D^n) \cap D'$ has full measure. (Note that we may also take the intersection of D with the interior of I^N, and thus avoid the details about the definition of the gradient on the boundary of I^N.)

Pick any $x \in D$. Since $u^n(x)$ is differentiable, $\nabla u^n(x)$ equals the unique subgradient at x (Rockafellar (1970), Theorem 25.1, page 242). Therefore for any y in I^N, $x \in D$,

$$\frac{u^n(x - \delta y) - u^n(x)}{\delta} \leq \nabla u^n(x) \cdot y \leq \frac{u^n(x + \delta y) - u^n(x)}{\delta},$$

for all small $\delta > 0$. (Note that for sufficiently small δ, the points $(x + \delta y) \in I^N$ and $(x - \delta y) \in I^N$).

We now show that for any $\epsilon > 0$, there is $\bar{n}$ such that $n > \bar{n}$ implies

$$\frac{u(x - \delta y) - u(x)}{\delta} - \epsilon \leq \nabla u^n(x) \cdot y \leq \frac{u(x + \delta y) - u(x)}{\delta} + \epsilon. \tag{3}$$

To see this, note that given any two sequence of real numbers r^n, s^n, with $r^n \geq s^n, \forall n$, and $s^n \longrightarrow s$, the following inequalities hold: $r^n - s \geq s^n - s \geq -||s^n - s||$. Since for any $\epsilon > 0$, there is $\bar{n}$ such that $n > \bar{n}$ implies $-||s^n - s|| \geq -\epsilon$, it follows that $[n > \bar{n} \implies r^n - s \geq -\epsilon]$. The same argument can be used to obtain both inequalities in (3).

Finally letting $\delta \downarrow 0$, by the definition of gradient (3) yields

$$\nabla u(x) \cdot y - \epsilon \leq \nabla u^n(x) \cdot y \leq \nabla u(x) \cdot y + \epsilon.$$

Since y and ϵ are arbitrary, this implies the desired result.

Q.E.D.

Lemma 6 *The set V of measures generated by functions of W is norm compact.*

Proof Let $\mu^n, n = 1, 2, \ldots$ be a sequence in V. Each $\mu^n \in V$ is defined by a function $u^n \in W$. By Lemma 4, a subsequence of u^n (which abusing notation we indicate with a subindex n) converges uniformly to some $u \in W$. By Lemma 5, in a further subsequence (also indicated with the same subindex n), ∇u^n converges pointwise λ-a.e. to ∇u. By Egoroff Theorem (see for instance, Aliprantis and Border (1999), page 349), for any $\epsilon > 0$, there is a set $G \subset I^N$ with $\lambda(G) > 1 - \epsilon$, such that ∇u^n converges uniformly to ∇u in G.

Thus

$$\begin{aligned}
\sup_E |\mu^n(E) - \mu(E)| &= |\int_E \{(\nabla u^n(x) - \nabla u(x)) \cdot x - (u^n(x) - u(x))\}\, d\lambda| \\
&\leq \int_E |(\nabla u^n(x) - \nabla u(x)) \cdot x|\, d\lambda + \int_E |\,(u^n(x) - u(x))|\, d\lambda \\
&= \int_{E\cap G} |(\nabla u^n(x) - \nabla u(x)) \cdot x|\, d\lambda + \int_{E\cap G^c} |(\nabla u^n(x) - \nabla u(x)) \cdot x|\, d\lambda \\
&\qquad + \int_E |\,(u^n(x) - u(x))|\, d\lambda
\end{aligned}$$

As n tends to infinity, the first and third terms of the last line go to zero. The integrand in the second term is bounded: since, $0 \leq \nabla u^n \cdot x \leq 1 \cdot x \leq N$ and the same bounds apply to $\nabla u \cdot x$, it follows that $|(\nabla u^n - \nabla u) \cdot x| \leq N$. Thus,

$$\int_{E\cap G^c} |(\nabla u^n(x) - \nabla u(x)) \cdot x|\, d\lambda \leq N\lambda(G^c) \leq N\epsilon.$$

Since ϵ is arbitrary, the desired result obtains.

Q.E.D.

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