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"A closed-form solution for defaultable bonds with log-normal spread"

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A closed-form solution for defaultable bonds with log-normal spread

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Abstract. In this paper we describe a two factor model for a defaultable discount bond, assuming a log-normal dynamics for the instantaneous short rate spread. Under some simplified hipothesis, we obtain an explicit barrier-type solution for zero recovery and constant recovery.

1 Introduction

The approaches to model credit risk can be broadly classified in two classes. The earlier includes the so called structural models, based on the firm's value approach introduced by Merton in [14] and extended by Black and Cox [2], Longstaff and Schwartz [13] and others. A review of the literature on these models can be seen in [20].

More recent is the class of the generally termed as reduced form models, in which the assumptions on a firm's value are dropped, and the default is modelled as an exogenous stochastic process. This class of models has been studied in [4], [5], [6], [7], [12], [11] and others. For a review we refer the reader to [7] and [20].

The goal in this paper is to describe a two factor model, in the context of the Black&Scholes option pricing techique, that could be applied sovereign defaultable bonds. The price of a risky bond price is derived as a function of the risk-free short rate and the instantaneous short spread, and the requirement is that the short spread must be positive. The dynamic of the spread is assumed to satisfy a log-normal diffusion with bounded volatility.

Our approach is connected with a remark in [20], saying that an alternative to the modeling of the term structure of defaultable bonds, based on the Heat-Jarrow-Morton (HJM) approach (cf. [9]), would be a two factor model using an arbitrage free model for the risk-free rate, and a model for the forward spread that generates a positive short rate spread.

The model in [3], applied to Brady bonds, was developed in the same context that ours, in the sense that they use the Black&Scholes pricing technique, but they took expectation

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on the risk of default instead of hedging it, so our pricing equation and its solution are different from theirs.

This paper is organized as follows. The bond pricing equation is derived in section 2. In Section 3 we obtain the solution for a lognormal dynamics of the short spread without recovery, and in Section 4 we consider a constant recovery. Section 5 contains the conclusions and some comments on possible future work.

2 The pricing equation

We work in a continuous time framework, in which $r_d(t)$ is the defaultable short rate if a default event has not ocurred until t, $r(t)$ is the risk-free short rate, and the spread $h(t)$ is defined as

$$h(t) = r_d(t) - r(t).$$

Our assumptions are

1. at any time t risk-free discount bonds and defaultable discount bonds of all maturities are available,
2. the dynamic of $r(t)$ and $h(t)$ are governed by diffusion equations

$$dr(t) = \mu_r(r,t)dt + \sigma_r(r,t)dW_1, \tag{1}$$

$$dh(t) = \mu_h(h,t)dt + \sigma_h(h,t)dW_2, \tag{2}$$

   where W_1 and W_2 are uncorrelated standard Brownian motions,
3. the spread $h(t) > 0$ is positive.

To derive a general equation for the defaultable bond, we set a portfolio Π containing a defaultable bond $P(r,h,t,T)$, of maturity T, a number Δ of risk free bonds $B(r,t,T_1)$, of maturity T_1, and a number Δ_1 of defaultable bonds $C(r,h,t,T_2)$ of maturity T_2,

$$\Pi = P(r,h,t,T) - \Delta B(r,t,T_1) - \Delta_1 C(r,h,t,T_2).$$

From Itô's lemma it follows that

$$d\Pi = \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_r(r,t)\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma_h(h,t)\frac{\partial^2 P}{\partial h^2}\right)dt + \frac{\partial P}{\partial r}dr\frac{\partial P}{\partial h}dh$$

$$\Delta\left[\left(\frac{\partial B}{\partial t} + \frac{1}{2}\sigma_r(r,t)\frac{\partial^2 B}{\partial r^2}\right)dt + \frac{\partial P}{\partial r}dr\right] -$$

$$-\Delta_1\left[\left(\frac{\partial C}{\partial t}+\frac{1}{2}\sigma_r(r,t)\frac{\partial^2 C}{\partial r^2}+\frac{1}{2}\sigma_h(h,t)\frac{\partial^2 C}{\partial h^2}\right)dt+\frac{\partial C}{\partial r}dr+\frac{\partial C}{\partial h}dh\right],$$

and we look for values of Δ and Δ_1 that eliminate the randomness in $d\Pi$.

$$\Delta_1=\frac{\frac{\partial P}{\partial h}}{\frac{\partial C}{\partial h}},\quad \Delta=\frac{1}{\frac{\partial B}{\partial r}}\left[\frac{\partial P}{\partial r}-\frac{\frac{\partial P}{\partial h}}{\frac{\partial C}{\partial h}}\frac{\partial C}{\partial r}\right]. \tag{3}$$

Using non arbitrage arguments it follows that

$$\mathcal{L}_1(P)-\Delta\mathcal{L}(B)-\Delta_1\mathcal{L}_1(C)=0, \tag{4}$$

where

$$\mathcal{L}()=\frac{\partial}{\partial t}+\frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}-r \quad\text{and}\quad \mathcal{L}_1()=\frac{\partial}{\partial t}+\frac{1}{2}\sigma_r^2\frac{\partial^2}{\partial r^2}+\frac{1}{2}\sigma_h^2\frac{\partial^2}{\partial h^2}-r.$$

Replacing Δ y Δ_1 in (4) we obtain

$$\frac{1}{\left(\frac{\partial P}{\partial r}\frac{\partial C}{\partial h}-\frac{\partial P}{\partial h}\frac{\partial C}{\partial r}\right)}\left[\frac{\partial C}{\partial h}\mathcal{L}_1(P)-\frac{\partial P}{\partial h}\mathcal{L}_1(C)\right]=\frac{\mathcal{L}(B)}{\frac{\partial B}{\partial r}},$$

where, as we know, the right hand side is

$$\frac{\mathcal{L}(B)}{\frac{\partial B}{\partial r}}=\lambda_r(r,t)\sigma_r(r,t)-\mu_r(r,t),$$

and $\lambda_r(r,t)$ is the market price of rate risk.

Then

$$\frac{\partial C}{\partial h}\mathcal{L}_1(P)-\frac{\partial P}{\partial h}\mathcal{L}_1(C)=(\lambda_r\sigma_r-\mu_r)\left(\frac{\partial P}{\partial r}\frac{\partial C}{\partial h}-\frac{\partial P}{\partial h}\frac{\partial C}{\partial r}\right)$$

Rewriting this equation as

$$\frac{\mathcal{L}_1(P)+(\mu_r-\lambda_r\sigma_r)\frac{\partial P}{\partial r}}{\frac{\partial P}{\partial h}}=\frac{\mathcal{L}_1(C)+(\mu_r-\lambda_r\sigma_r)\frac{\partial C}{\partial r}}{\frac{\partial C}{\partial h}},$$

it can be seen that the ratio must be independent of the maturity, and hence equal to a quantity dependent of h and t, and possibly of r.For a given $\mu_h(h,t)$ and $\sigma_h(h,t)\neq 0$, it is always possible to write

$$\frac{\mathcal{L}_1(P) - (\mu_r - \lambda_r \sigma_r)\frac{\partial P}{\partial r}}{\frac{\partial P}{\partial h}} = \lambda_h(r,h,t)\sigma_h(h,t) - \mu_h(h,t)$$

where $\lambda_h(r,h,t)$ is the market price of the risk associated with the spread. We shall assume in what follows that λ_h does not depend on r. To shorten the notation we set

$$\phi(r,t) = \mu_r(r,t) - \lambda_r(r,t)\sigma_r(r,t), \qquad \phi(r,t) = \mu_h(h,t) - \lambda_h(h,t)\sigma_h(h,t)$$

for the adjusted drifts of interest rate and spread, respectively.

Writing $\mathcal{L}_1(P)$ explicitly, we arrive to the pricing equation of the defaultable bond

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_r^2(r,t)\frac{\partial^2 P}{\partial r^2} + \frac{1}{2}\sigma_h^2(h,t)\frac{\partial^2 P}{\partial h^2} + \phi(r,t)\frac{\partial P}{\partial r} + \psi(h,t)\frac{\partial P}{\partial h} - rP = 0. \tag{5}$$

As long as r and h were not correlated, the problem is separable; i.e. we consider a solution

$$P(r,h,t,T) = Z(r,t,T)S(h,t),$$

where $Z(r,t,T)$ is the solution of a risk free bond [1]. Replacing this solution in (5) gives

$$Z\left[\frac{\partial S}{\partial t} + \frac{1}{2}\sigma_h^2(h,t)\frac{\partial^2 S}{\partial h^2} + \psi(h,t)\frac{\partial S}{\partial h}\right] + S\left[\frac{\partial Z}{\partial t} + \frac{1}{2}\sigma_r^2(r,t)\frac{\partial^2 Z}{\partial r^2} + \phi(r,t)\frac{\partial Z}{\partial r} - rZ\right] = 0.$$

Then $S(h,t)$ satisfies

$$\frac{\partial S}{\partial t} + \frac{1}{2}\sigma_h^2(h,t)\frac{\partial^2 S}{\partial h^2} + \psi(h,t)\frac{\partial S}{\partial h} = 0. \tag{6}$$

If default has not occurred before the maturity T, the final condition is

$$P(r,h,T,T) = Z(r,T,T)S(h,T) = 1,$$

which leads to the following final conditions for Z and S

$$Z(r,T,T) = 1, \quad S(h,T) = 1.$$

3 Modeling the spread without recovery

The log-normal assumption for the dynamics of $h(t)$ is the natural and simplest way to assure its positivity. In [10] and [16] has been shown that this assumption is not suitable for continuously compounded interest rates, since it implies that the rates explode with positive probabilities, therefore expected accumulation factors over any finite time interval

[1] For a full description of interest rate models see [17].

are infinite. This problem has been addressed e.g., in [8], [18], and [15], where alternative log-normal type term structures that preclude explosion of rates are proposed.

However, for a log-normal term structure model, the spread is positive and remains finite. The spread increases as it becomes (or it is perceived to become) more likely that the bond may default. But it does not rise unboundedly; in the practice there is a finite upper barrier, even if it is not known in advance.

For a log normal diffusion, imposing an upper bound to the short spread, $0 \leq h \leq H_d < \infty$, is equivalent to define a bounded volatility process, i.e.

$$dh(t) = \mu_h(h,t)dt + \sigma(h,t)dW_2, \tag{7}$$

with

$$\sigma_h(h,t) = \min(H_d, h(t))\sigma_h(t), \tag{8}$$

where $\sigma_h(t)$ is a deterministic function, and it is shown in [9] that this volatility process gives finite positive rates (spread in this case).

For this first version of the model we shall make some simplified assumptions that allow us to easily obtain a closed-form solution:

1. $\lambda_h = \lambda_0$ and $\sigma_h(t) = \sigma_0$ are a positive constants.
2. $\mu_h(h,t) = \mu_0 h(t)$, where μ_0 is a positive constant.

With the above choices, equation (6) reduces to

$$\frac{\partial S}{\partial t} + \frac{1}{2}\sigma_0^2 h^2 \frac{\partial^2 S}{\partial h^2} + [\mu_0 - \lambda_0\sigma_0]\, h\frac{\partial S}{\partial h} = 0, \quad 0 \leq t < T, \quad 0 < h < H_d \tag{9}$$

with the final condition

$$S(h,T) = 1. \tag{10}$$

if default has not ocurred until maturity T.

Requiring that, for spread tending to zero, $P(t,h,t,T)$ should approximate to the solution of a risk-free discount bond, gives us the first boundary condition, namely

$$\lim_{h\to 0} S(h,t) = 1. \tag{11}$$

The second boundary condition, to be applied at H_d, arises from the assumption that a default occurs if ever h reaches H_d. Therefore, for zero recovery we must have

$$P(r, H_d, t, T) = 0, \tag{12}$$

which implies $S(H_d, t) = 0$.

With the usual change of variables

$$h = e^x, \quad t = T - \frac{2\tau}{\sigma_0^2}, \quad S(h,t,T) = e^{\alpha x + \beta\tau} u(x,\tau), \tag{13}$$

and for

$$\alpha = -\frac{1}{2}(k-1), \quad \beta(\tau) = -\frac{1}{4}(k-1)^2\tau,$$

the problem (9) becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \tau > 0, \quad -\infty < x < \ln H_d, \tag{14}$$

with initial condition

$$u(x,0) = e^{\frac{1}{2}(k-1)x}, \tag{15}$$

and boundary conditions

$$u(\ln H_d, \tau) = 0,$$

$$\lim_{x \to -\infty} u(x,\tau) = e^{\frac{1}{2}(k-1)x + \frac{1}{2}(k-1)^2}.$$

Notice that the upper bound to the spread makes this problem mathematically similar to an up-and-out barrier option.
The solution to (14), obtained by the method of images, is (see Apendix A)

$$u(x,\tau) = e^{\frac{1}{4}(k-1)^2\tau}\left[e^{\frac{1}{2}(k-1)x}N(d_1) - e^{\frac{1}{2}(k-1)(2\ln H_d - x)}N(d_2)\right],$$

where

$$d_1(x,\tau) = \frac{\ln H_d - x}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau},$$

$$d_2(h,t,T) = \frac{x - \ln H_d}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}$$

is the cumulative probability distribution function for a normally distributed variable with mean zero and variance 1.

Going back to (13), we can write the solution in financial variables

$$S(h,t) = N(d_1) - e^{(k-1)(\ln H_d - \ln h)}N(d_2), \tag{16}$$

where

$$d_1(h,t) = \frac{\ln\left(\frac{H_d}{h}\right)}{\sqrt{2(T-t)}} - \frac{1}{2}(k-1)\sqrt{2(T-t)},$$

$$d_2(h,t) = \frac{\ln\left(\frac{h}{H_d}\right)}{\sqrt{2(T-t)}} - \frac{1}{2}(k-1)\sqrt{2(T-t)}.$$

It is easy to see that the final condition and the boundary condition at H_d are verified by construction.
For $t = T$, $N(d_1) = 1$ and $N(d_2) = 0$. Hence $S(h,T) = 1$.

At $h = H_d$, $d_1 = d_2 = -\frac{1}{2}(k-1)\sqrt{2(T-t)}$, wich yields $S(H_d, t) = 0$.

It remains to check the boundary condition for $h \longrightarrow 0$,

$$\lim_{h\to 0} S(h,t) = \lim_{h\to 0}\left[N(d_1) - e^{(k-1)(\ln H_d - \ln h)}N(d_2)\right].$$

Since for $h \longrightarrow 0$, $d_1 \longrightarrow \infty$, then $\lim_{h\to 0} N(d_1) = 1$.

To calculate the limit of the second term in the bracket we use the assymptotic expression for the cumulative normal probability distribution function (see [1]). For $x < 0$

$$N(x) = \frac{Z(x)}{x}\left[1 - \frac{1}{x^2} + \frac{1.3}{x^4} + \frac{1.3.5}{x^4} + \ldots \frac{(-1)^n.1.3\cdots(2n-1)}{x^{2n}}\right] + R_n,$$

where

$$Z(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},$$

and R_n is less in absolute value then the first neglected term. It results (see Appendix B)

$$\lim_{h\to 0} N(d_2) = 0.$$

Therefore, $\lim_{h\to 0} S(h,t) = 1$.

4 Modeling the spread with constant recovery

Introducing a recovery is equivalent to specify a boundary condition

$$S(H_d, t) = R(t),$$

and, due to this contribution, there will be an extra term $S_R(h,t)$ added to the solution (16).

For the particular case of a recovery paid in cash, or when the recovery is a fraction of the face value, $R(t) = R$ is constant; this makes the problem mathematically equivalent to the modeling of a constant rebate for and up-and-out barrier. The additional term takes the form (see Appendix C)

$$S_R(h,t) = e^{-\frac{1}{2}(k-1)\ln h - \frac{1}{4}(k-1)^2(T-t)} R\left[1 - \operatorname{erf}(d_3)\right],$$

where

$$d_3 = \frac{\ln H_d - \ln h}{2\sqrt{T-t}}, \quad \text{and} \quad \operatorname{erf}(t) = \frac{1}{\sqrt{\pi}} \int_0^t e^{-\rho^2} d\rho$$

5 Conclusions

Under some simplified assumptions we have obtained a barrier type closed-form solution for a two factor model of a defaultable bond, modeling the spread as a log-normal random walk with bounded volatility.

This log-normal type model for the spread is the simplest one, and by relaxing the hypothesis it may be improved to better agree with some phenomenological facts. In [6] is pointed out that the behaviour of instantaneous risk of default was observed to be mean reverting under the real measure. Therefore, out next step shall be to consider a mean-reverting lognormal type random walk, and preliminary calculations show that, in this case, a quasi-closed solution may be obtained in terms of the Hermite polynomials.

A jump diffusion model should also be considered for the spread. Allowing jumps would be a more realistic assumption, especially for certain sovereign issuers, where it is likely that political events may produce jumps in the spread.

6 Appendix A

Consider the problem (9), (10), (11), and (12).
Setting

$$h = e^x, \quad t = T - \frac{2\tau}{\sigma_0^2}, \quad S(h,t) = v(x,\tau),$$

and replacing this change of variables in (9) one obtains

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x}, \quad \tau > 0, \quad -\infty < x < \ln H_d. \tag{17}$$

with initial and boundary conditions

$$v(x,0)=1, \quad v(\ln H_d,\tau)=0, \quad \lim_{x\to-\infty} v(x,\tau)=1.$$

Now we set

$$v(x,\tau)=e^{\alpha x+\beta tau}u(x,\tau) \tag{18}$$

and (17) becomes

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}+[2\alpha+(k-1)]\frac{\partial u}{\partial x}+\left[\alpha^2+\alpha(k-1)-\beta\right]u. \tag{19}$$

The choice

$$\alpha=-\frac{1}{2}(k-1), \quad \beta=-\frac{1}{4}(k-1)^2\tau \tag{20}$$

eliminates the terms in $\frac{\partial u}{\partial x}$ and u, thus reducing (19) to the heat equation in a semi-infinite domain

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}, \quad \tau>0, \quad -\infty<x<\ln H_d, \tag{21}$$

with initial condition

$$u(x,0)=e^{\frac{1}{2}(k-1)x},$$

and boundary conditions

$$u(\ln H_d,\tau)=0$$

$$\lim_{x\to-\infty} u(x,\tau)=e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2}$$

The well known general solution this problem, using the method of images, is

$$u(x,\tau)=\int_{-\infty}^{\ln H_d} u_0(y)\left[G(x-y,\tau)-G(x-(2\ln H_d-y)\tau)\right]dy, \tag{22}$$

where

$$G(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}e^{-\frac{x^2}{4\tau}} \tag{23}$$

is the fundamental solution for the heath equation.
Replacing (23) in (22) we have

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\left[\int_{-\infty}^{\ln H_d} u_0(y)e^{-\frac{(x-y)^2}{4\tau}}dy-\int_{-\infty}^{\ln H_d} u_0(y)e^{-\frac{(x-2\ln H_d+y)^2}{4\tau}}dy\right]=I_1-I_2$$

where u_0 is given by (15).

Here

$$I_1 = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln H_d} u_0(y) e^{-\frac{(x-y)^2}{4\tau}} dy. \tag{24}$$

Substituting

$$z = \frac{y-x}{\sqrt{2\tau}}$$

in I_1 yields

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln H_d - x}{\sqrt{2\tau}}} e^{\frac{1}{2}(k-1)x} e^{\frac{1}{2}(k-1)\sqrt{2\tau}z-} e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{\frac{1}{2}(k-1)x}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln H_d - x}{\sqrt{2\tau}}} e^{-\frac{1}{2}\left[z^2-(k-1)\sqrt{2\tau}z)\right]} dz$$

$$= \frac{e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln H_d - x}{\sqrt{2\tau}}} e^{-\frac{1}{2}\left[z-\frac{1}{2}(k-1)\sqrt{2\tau}\right]^2} dz.$$

Calling

$$\rho = z - \frac{1}{2}(k-1)\sqrt{2\tau}$$

we obtain

$$I_1 = \frac{e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln H_d - x}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau}} e^{-\frac{\rho^2}{2}} d\rho,$$

$$= e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k-1)^2\tau} N(d_1) \tag{25}$$

where

$$d_1(x,\tau) = \frac{\ln H_d - x}{\sqrt{2\tau}} - (k-1)\sqrt{2\tau},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}x^2} dx,$$

is the cumulative probability distribution function for a normally distributed variable with mean zero and variance 1.
The calculation of I_2 is similar to that of I_1.

$$I_2 = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\ln H_d} u_0(y) e^{-\frac{(x-2\ln H_d+y)^2}{4\tau}} dy. \tag{26}$$

Putting

$$z = \frac{x - 2\ln H_d + y}{\sqrt{2\tau}},$$

we obtain

$$I_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln H_d}{\sqrt{2\tau}}} u_0(2\ln H_d - x + \sqrt{2\tau}z) e^{-\frac{z^2}{2}} dz,$$

where we substitute

$$u_0(\sqrt{2\tau}z - x + 2\ln H_d) = e^{\frac{1}{2}(k-1)\left[\sqrt{2\tau}z - x + 2\ln H_d\right]}$$

to obtain

$$I_2 = \frac{e^{\frac{1}{2}(k-1)(2\ln H_d - x)}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln H_d}{\sqrt{2\tau}}} e^{-\frac{1}{2}\left[z^2 - (k-1)\sqrt{2\tau}z\right]} dz$$

$$= \frac{e^{\frac{1}{2}(k-1)(2\ln H_d - x) - + \frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln H_d}{\sqrt{2\tau}}} e^{-\frac{1}{2}\left[z - \frac{1}{2}(k-1)\sqrt{2\tau}\right]^2} dz.$$

The change of variables

$$\rho = z - \frac{1}{2}(k-1)\sqrt{2\tau}, \quad d\rho = dz,$$

gives

$$I_2 = \frac{e^{\frac{1}{2}(k-1)(2\ln H_d - x) - + \frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-\ln H_d}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau}} e^{-\frac{\rho^2}{2}} d\rho$$

$$= e^{\frac{1}{2}(k-1)(2\ln H_d - x) - + \frac{1}{4}(k-1)^2\tau} N(d_2), \qquad (27)$$

where

$$d_2 = \frac{x - \ln H_d}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau}.$$

Finally, from (25) y (27) we have

$$u(x,\tau) = e^{\frac{1}{4}(k-1)^2\tau -} \left[e^{\frac{1}{2}(k-1)x} N(d_1) - e^{\frac{1}{2}(k-1)(2\ln H_d - x)} N(d_2)\right]$$

Going back to (18) we can write

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x} \left[e^{\frac{1}{2}(k-1)x} N(d_1) - e^{\frac{1}{2}(k-1)(2\ln H_d - x)} N(d_2)\right]$$

$$= N(d_1) - e^{(k-1)(\ln H_d - x)} N(d_2)$$

7 Appendix B

To calculate

$$\lim_{x\to-\infty} e^{(k-1)(\ln H_d - x)} N(d_2)$$

we use the assymptotic expression for the cumulative normal distribution. For $x < 0$

$$N(x) = \frac{Z(x)}{x}\left[1 - \frac{1}{x^2} + \frac{1.3}{x^4} + \frac{1.3.5}{x^4} + \cdots \frac{(-1)^n.1.3\cdots(2n-1)}{x^{2n}}\right] + R_n,$$

where

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

and R_n is less in absolute value then the first neglected term. Then

$$\lim_{x\to-\infty} e^{(k-1)(\ln H_d - x)} \frac{e^{-\frac{d_2^2}{2}}}{d_2}\left[1 - \frac{1}{d_2^2} + \frac{1.3}{d_2^4} + \frac{1.3.5}{d_2^4} + \cdots \frac{(-1)^n.1.3\cdots(2n-1)}{d_2^{2n}}\right]$$

$$= \lim_{x\to-\infty} e^{(k-1)(\ln H_d - x)} \frac{e^{-\frac{d_2^2}{2}}}{d_2} = \tag{28}$$

$$= \lim_{x\to-\infty} \frac{\exp\left[(k-1)(\ln H_d - x) - \frac{1}{2}\left[\frac{(x-\ln H_d)^2}{2\tau} - (k-1)(x - \ln H_d) + \frac{1}{4}(k-1)^2 2\tau\right]\right]}{d_2}$$

$$= \lim_{x\to-\infty} \frac{\exp\left[-\frac{1}{2}\left[\frac{x-\ln H_d}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\right]^2\right]}{\frac{x-\ln H_d}{\sqrt{2\tau}} - \frac{1}{2}(k-1)\sqrt{2\tau}} = 0$$

8 Appendix C

The problem (14), with a specified boundary condition at $x = \ln H_d$

$$u(\ln H_d, \tau) = g(\tau)$$

can be reduced to two simpler problems with solutions u_1 and u_2 such that $u = u_1 + u_2$. The subproblem for u_1 is the already solved model with zero recovery. The second subproblem is

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial x^2}$$

$$u_2(x, 0) = 0$$

$$u_2(\ln H_d, \tau) = R(\tau)$$

$$\lim_{x \to -\infty} u_2(x, \tau) = 0$$

Taking the Laplace transform with respect to τ, we get the ordinary differential equation

$$\frac{\partial^2 \hat{u}2}{\partial x^2} = p\hat{u_2} \tag{29}$$

where

$$\hat{u}_2(x,p) = \int_0^\tau u(x,\tau)e^{-p\tau}d\tau$$

is the Laplace transform of $u(x, \tau)$.
The solution to (29) is

$$\hat{u}_2(x,p) = \hat{q}(p)e^{\sqrt{p}(x-\ln H_d)}$$

where $\hat{R}(p)$ is the Laplace transform of $R(\tau)$. Therefore, $u(x, \tau)$ can be written as the Laplace convolution of $R(\tau)$ with the inverse Laplace transform of $e^{\sqrt{p}(x-\ln H_d)}$

$$u_2(x,\tau) = \frac{1}{2\sqrt{\pi}}\int_0^\tau q(u)(x - \ln H_d)\exp\left[-\frac{(x-\ln H_d)^2}{4(\tau-u)}\right]\frac{1}{(\tau-u)^{\frac{3}{2}}}du$$

Setting $y = \frac{1}{\tau - u}$, and for $R(\tau) = R = cte$, we obtain

$$u_2(x,\tau) = \frac{R(x-\ln H_d)}{2\sqrt{\pi}}\int_{\frac{1}{\tau}}^\infty \exp\left[\frac{-y(x-\ln H_d)^2}{4}\right]\frac{1}{\sqrt{y}}dy$$

$$= \frac{R}{\sqrt{\pi}}\int_{\frac{x-\ln H_d}{\sqrt{\tau}}}^\infty e^{-\frac{z^2}{4}}dz$$

$$= R\left[1 - \operatorname{erf}\left(\frac{x-\ln H_d}{\sqrt{\tau}}\right)\right],$$

where

$$\operatorname{erf}(t) = \frac{1}{\sqrt{\pi}}\int_0^t e^{-\rho^2}d\rho$$

is the error function.

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