QUAERERE VERUM

UNIVERSIDAD DE SAN ANDRÉS

Seminario del Departamento de Economía

"Asset Market and the Consumption Dynamics with Boundedly Rational Investors."

Enrique Kawamura
(Universidad de San Andrés)

Martes 23 de mayo de 2000
11 hs.

Aula Chica de Planta Baja

Asset Market and Consumption Dynamics with Boundedly Rational Investors*

Enrique Kawamura
Universidad de San Andrés†

February 14th, 2000

Abstract

This paper is the first attempt of applying the Easley Rustichini (*Econometrica*, 1999, 5) replicator dynamics framework to security markets. With just the simplest one-period asset, traders choose portfolio with rules given by preferences over actions. Those preferences usually depend on prices but are subject to evolution through time. This preference transitions depend on past cosumption streams. I present some characterization of asset prices in the two action cases. I also discusss the problems of two action economies in the long run. Two examples are presented. Two lessons can be gotten. First, even with very simple (objective) state spaces, and with straightforward replicator dynamics governing preference evolution, prices can have very complex dynamics. In particular some type of chaotic behavior can be obtained. Second, even with a very simple definition of long run competitive equilibrium, the economy may not converge to it. This shows that this dynamics does not necessarily converge (in distribution) to any familiar competitive equilibrium concept. Preference evolution as well as date 0 preference on actions affect mainly this result.

1 Introduction

Bounded rationality has been the object of intense study during the last two decades. Several different lines have been taken by the literature. Traditionally in dynamic contexts the learning problem has been the focus of several

*Very Preliminary. Please Do Not Quote.

†Address: Vito Dumas 284, Victoria (1644), Buenos Aires, Argentina. Phone: 541-1-4725-7077. E-mail: kawa@udesa.edu.ar

studies, both at the microeconomic level [1] and to attack macroeconomic issues[2]. The traditional approach has been based on the standard Savage theory of uncertainty. Rational or Bayesian learning has been well developed. An important point is that it is the only type of learning consistent with the Savage axioms. However, many experimental and empirical studies have emphasized the failure of many of the predictions of models based on Savage states[3]. In complex environments foreseeing all states in the future may be completely non-sensible. For example, asset markets imply too many possible contingencies in the future to be taken into account (even to be imagined!). Some of them may be relevant enough to change portfolio decisions if considered in advance. However this is often not possible.

Modica and Rustichini ([8] and [9]) developed epistemic conditions for modelling unawareness in *static* frameworks. A second recent line of research, related to the unforeseen contingencies story, is the paper by Easley and Rustichini. In their work an individual agent chooses sequentially actions according to some rule, given by preferences over those actions. In each period, after payoffs are observed, those preferences evolve according to some transition that depends on realized payoffs. They show that under suitable conditions imposed on transitions actions converge to expected maximizing utility ones. This is really a decision theoretic foundation of adaptive learning, not based on Savage axioms. It is an important piece of work, filling a gap in the foundations of many ad-hoc adaptive learning devices used mainly in macroeconomics.

The point of the present paper is to give a first attempt of application of the Easley-Rustichini framework to an asset market. It seems reasonable to apply this model on security markets because it is in these where this type of adaptive learning could be observed. As stated above, contingencies present in asset markets are hard to be considered ex-ante in its totality. In fact casual observations of how trading is done resembles what happens in [5]. Hence in my paper agents will choose asset holdings according to some rules (given again by preferences over lotteries on actions). However those preferences will now depend on prices. This means that the rule by which a trader chooses the lottery depends on the price of the security. The intuition is as follows. If the security is very cheap, all traders would like to buy as much as possible. This is what usually happens when market considers a security *undervalued*. On the contrary, when the price is too high, traders would like to sell short the asset, resembling situations when the market tends

[1] See [6] for different aspects of learning in games

[2] See [11] for a survey of work in adaptive learning and macroeconomics.

[3] See [10] for a brief survey on this evidence as well as alternative ways of modelling bounded rationality.

to consider it *overvalued.* This is the idea behind the fact that preferences on actions depend on prices. It is also a convenient way to ensure market clearing.

The type of asset considered here is the simplest one. It is a short lived security that is traded at the beginning of each period and pays off a certain amount of the consumption good depending on the state. After this the trader consumes whatever is left. In the subsequent period the new endowment is again used to trade in the new asset market. In some sense the markets are repeated through time, avoiding wealth links among periods. The issue of wealth dynamics is not considered in this paper due to its additional complexity. One of the main problems with wealth dynamics in this context is the possible permanent bankruptcy of the agent. Since this is a first attempt to apply this type replicator dynamics to asset markets, I leave this problem for future research.

The main lesson that can be obtained is the following. Even though the number of actions can be small (for example, even if the trader can only sell short one unit, buy one unit or do nothing with the asset in each period), two states, and even if the replicator dynamics follow all the assumptions in [5], the dynamics of prices can be very complex. The reason is that prices can follow some type of combined dynamics that includes e.g. chaos. The first example in section 4 shows that. Another point of the paper is that although the dynamical system is usually ergodic, the long run behavior of the economy may not be rationalized as a stationary *more standard* type of competitive equilibrium definition. This depends on the date 0 preferences as well as the replicator dynamics. The definition of stationary long run equilibrium in section 4 resembles in some way some sort of *sunspot* equilibrium (in the sense of Cass and Shell). This is because in this equilibrium all the agents coordinate (exogenously given by the ergodic measure) to a certain long run security price. This is not necessarily linked directly to the realization of the states of nature.

Section 2 gives the basics of the model. Section 3 presents results of two action economies and discusses the problems of long run equilibria in this context. Section 4 presents two examples of three action economies, as well as the definition of the stationary long run equilibrium. It also discusses convergence (in distribution) of the economy to the stationary long run equilibrium. Section 5 gives concluding remarks and gives directions for extensions and future research.

2 The Model

The economy is populated by a continuum of ex-ante identical agents. These can be interpreted as investors. Time is discrete and goes to infinity. At each date a state of the economy is realized. To make this simple, assume that the state space is $S = \{1, 2\}$. There is one perishable consumption good, called *money*. Each investor receives an endowment $\bar{W}$ of money at the beginning of each period.

Each period there is a one-date asset. This is traded at the beginning of period t at a certain price q_t and its value is public information. The security pays off $(s_t - 1)$ units of money per unit of asset at the end of date t. The realization of s_t occurs after the date t market is closed. Once realized it is publicly observed. The asset is in zero net supply. There is no other security between periods.

Investors choose actions. Each action represents units sold short or bought of the asset. Assume that the action space is finite. Let A denote the action space with $A \equiv \{a^0, a^1, ..., a^N\}$. Following [5] investors are not fully rational. They do not have preferences on a stream of money through time. Instead, agents are born at date 0 with a certain period 0 preferences on (lotteries over) A. The preference relation depends upon the market price of the asset. Let $\succeq_0 |_q$ be the preference relation at date 0 given that the observed price of the asset is q. We make the following assumption borrowed from [5] in order to have a VonNeumann representation of $\succeq_0 |_q$.

Assumption 1 For each $q \geq 0$ the preference order $\succeq_0 |_q$ is a weak order; that is:

1.i For all γ, δ in $\Delta(A)$ either $\gamma (\succeq_0 |_q) \delta$ or $\delta (\succeq_0 |_q) \gamma$

1.ii For all γ, δ, η in $\Delta(A)$, if $\gamma (\succeq_0 |_q) \delta$ and $\delta (\succeq_0 |_q) \eta$ then $\gamma (\succeq_0 |_q) \eta$.

It also satisfies the independence and the continuity axioms for every q (see assumption WIC in [5] for the formal definition of these).

It satisfies continuity on q in the closed convergence topology. This means that if $q_n \to q$ then $(\succeq_0 |_{q_n}) \to (\succeq_0 |_q)$ and this satisfies the three axioms above.

Assume finally that, for all $n \geq 1$, $a^n (\succ_0 |_q) a^0$ (action n dominates action 0 strictly).

In our case, action a^0 can be interpreted as a very large short sale amount, or very large purchase amount, so that investors never find optimal to take that action. With this assumption, the relation $\succeq_0 |_q$ can be represented by the following formula. Let γ, ϕ be on $\Delta(A)$. Then

$$\gamma (\succeq_0 |_q) \phi \quad \Leftrightarrow \quad \sum_{n=0}^{N} u_0^n(q) \gamma_n \geq \sum_{n=0}^{N} u_0^n(q) \phi_n$$

where $(u_0^n(q))_{n=0}^N$ are continuous functions of q. Since action 0 is always dominated then we can define

$$v_0^n(q) \equiv \frac{u_0^n(q) - u_0^0(q)}{\sum_{n=1}^N (u_0^n(q) - u_0^0(q))}$$

Recall that $v_0^n(q)$ can be interpreted as the relative weight of action n that each agent chooses. In fact, in the equilibrium (to be defined below) it is the proportion of agents who actually takes action n. We see that $v_0^n(q) \in int\Delta(A)$.

Without loss of generality, we can label all actions $a^1, a^2, ..., a^N$ such that $a^1 < a^2 < ... < a^N$. Let us define $A^* \equiv \{a^1, ..., a^N\}$. Therefore, in order to ensure existence of equilibrium I impose the following assumption.

Assumption 2 The vector-valued function $v_0(q) \equiv [v_0^1(q), ..., v_0^N(q)]^T$ is such that $\lim_{q \to 0} v_0(q) = [0, 0, ...0, 1]^T \in \Re_+^N$ and $\lim_{q \to \infty} v_0(q) = [1, 0, ...0, 0]^T \in \Re_+^N$.

The idea behind this is simple. If the asset were free, then everybody would try to purchase as much as possible. If the asset is infinitely costly, then everybody would intend to sell short the asset as much as possible. Although I do not give a foundation for this, it is still quite intuitive.

The timing within date 0 is as follows. First the agent chooses an action a^n using its period 0 preference relation. This is a portfolio choice problem. In principle we demand that the following budget constraint holds:

$$\sum_{n=1}^{N} a^n q \leq \bar{W}$$

This in general will not be binding. Then the state s_0 is realized. After delivery of goods is observed according to the payoff of the asset and the position of the agent (whether she is long or short in the security). The final amount of money is consumed by the agent at the end of period 0.

The equilibrium concept for the first period is the following.

Note that this equilibrium definition is independent of what is the realization of s_0. The reason is that decisions taken by the investors only depend on the market price. Since preference only depend on prices and actions, they do not take into account the possible values of s_0. In this sense the choice is done without any belief (probability distribution) defined on S.

After the observation of s_0, preferences evolve through a transition, denoted by T. This maps elements in the set $R \times S$ onto R. This is the same as in [5]. The only difference again is that the new preference order is function of q. Given the VN representation of preferences we can state that $T\left[\left(\succeq_0 |_q\right), s\right] \equiv F\left(v_0(q), s\right) \equiv v_1(q)$. I impose the same restrictions on $v_1(q)$ as on $v_0(q)$. Then q_1^* is obtained through date 1 market clearing:

$$\sum_{n=1}^{N} a^n v_1^n\left(q_1^*\right) = 0$$

This is the date 1 boundedly rational date 1 equilibrium. After s_1 is realized, the same T map gives $v_2(q)$ and the process is replicated to infinity. Then it is possible to define a boundedly rational equilibrium.

Definition 1 *A boundedly rational equilibrium date-0-price is a price q_0^* such that at that given price the asset market clears, i.e.,*

$$\sum_{n=1}^{N} a^n v_0^n\left(q_0^*\right) = 0$$

I need to ensure that this equilibrium concept is not vacuous. This is confirmed using the assumptions 1 and 2 above together with a suitable law of motion for preferences. Easley and Rustchini [5] have shown that the transition that ensures convergence to objective maximizing utility actions (in the one agent problem case) takes the following form.

$$\frac{v_{t+1}^1}{v_{t+1}^n} = \left(\frac{v_t^1(q)}{v_t^n(q)}\right)\left(\frac{f\left(c_t^1\left(s^t\right)\right)}{f\left(c_t^n\left(s^t\right)\right)}\right) \tag{1}$$

where $n = 2, ..., N$, for some strictly increasing, strictly positive function f. Here I define

$$c_t^n\left(s^t\right) \equiv \max\left[\bar{W} - a^n q_t^* + a^n\left(s_t - 1\right); 0\right]$$

where q_t^* is the beginning-of-period t price of the security. Note then that I do not attempt to obtain evolution of preferences *out of equilibrium path*. In other words, the evolution will depend entirely on the equilibrium price. The main reason for this is to avoid preference dynamics depending on the whole history of prices. Given this the following results is easy to show.

Proposition 1 *If $a^1 < 0$ and $a^N > 0$ then the set of boundedly rational equilibrium date 0 prices is non empty. Moreover if $N = 2$, and if $v_0^1(q)$ is strictly increasing the equilibrium price at date 0 is unique.*

P roof. Existence is ensured in period 0 due to assumption 2. For any other period t, I claim that assumption 2 can be also extended to any date t, given the law of motion in equation (1). The proof of this claim is by induction. In period $t+1$ weights are given by the following equation.

$$v_{t+1}^n(q) = \frac{v_t^n(q) f(c_t^1(s^t))}{\sum_{n=1}^N v_t^n(q) f(c_t^n(s^t))} \tag{2}$$

Then, for $t = 0$ we see that

$$v_1^n(q) = \frac{v_0^n(q) f(c_0^1(s_0))}{\sum_{n=1}^N v_0^n(q) f(c_0^n(s^0))}$$

Since $f > 0$, by assumption 2, we know that

$$\begin{aligned}\lim_{q\to 0} v_1^N(q) &= \lim_{q\to 0} \frac{v_0^N(q) f(c_0^1(s^0))}{\sum_{n=1}^N v_0^n(q) f(c_0^n(s^0))} \\ &= \lim_{q\to 0} \frac{v_0^N(q)}{v_0^N(q)} = \frac{1}{1} = 1\end{aligned}$$

and of course for any other n,

$$\begin{aligned}\lim_{q\to 0} v_1^n(q) &= \lim_{q\to 0} \frac{v_0^n(q) f(c_0^1(s^0))}{\sum_{n=1}^N v_0^n(q) f(c_0^n(s^0))} \\ &= \frac{0}{f(s^{N,0})} = 0\end{aligned}$$

Similarly:

$$\begin{aligned}\lim_{q\to \infty} v_1^1(q) &= \lim_{q\to \infty} \frac{v_0^1(q) f(c_0^1(s^0))}{\sum_{n=1}^N v_0^n(q) f(c_0^n(s^0))} \\ &= \lim_{q\to \infty} \frac{v_0^1(q)}{v_0^1(q)} = \frac{1}{1} = 1\end{aligned}$$

and for any other t, $\lim_{q\to\infty} v_1^n(q) = 0$. Using the inductive principle the same holds for any other period $t \geq 2$. Hence existence is also confirmed for any other period t.

Suppose that $N = 2$. In period 0, uniqueness follows from the standard intermediate value theorem due to monotonicity of $v_0^1(q)$. Then

$$v_1^1(q) = \frac{v_0^1(q)\, f\left(c_0^1\left(s^0\right)\right)}{v_0^1(q)\left[f\left(c_0^1\left(s^0\right)\right) - f\left(c_0^2\left(s^0\right)\right)\right] + f\left(c_0^2\left(s^0\right)\right)}$$

It should be clear that $v_1^1(q)$ is strictly increasing in q. For if $q_A < q_B$ then $v_0^1(q_A) < v_0^1(q_B)$. But the fraction

$$\frac{ax}{bx + c}$$

with $a > 0$ and $c > 0$ is strictly increasing in x. Hence the right hand side evaluated at q_A is strictly less than when evaluated at q_B. Using inductive arguments it can be shown that $v_t^1(q)$ is strictly increasing in q. Then the equilibrium price process q_t^* is unique. This completes the proof. ■

I next study the two action case. This is a simple way to visualize the dynamics of the equilibrium price, as well as of the individual wealth.

3 Two action economies. Preliminary results.

In this section it is assumed that $N = 2$. One of the reasons of concentrating the analysis in this case is the uniqueness result. In order to characterize dynamics of equilibria, it is simpler if we only have a unique process. Otherwise some kind of equilibrium selection should be provided. Since this is beyond the scope of this paper, I focus on the case where unique equilibrium price process is ensured to exist.

The first result concerns the dynamics of weight through time. This is a straightforward extension of [5].

Proposition 2 *Assume equation* (1) *Then equilibrium weights in the two action case have the following form:*

$$v_{t+1}^1\left(q; s^t\right) = \frac{v_0^1(q)}{v_0^1(q)\left[1 - \Lambda_t\left(s^t\right)\right] + \Lambda_t\left(s^t\right)}$$

where

$$\Lambda_t\left(s^t\right) \equiv \prod_{\tau=0}^{t} \phi_\tau\left(s^\tau\right)$$

and where

$$\frac{1}{\phi_t\left(s^t\right)} \equiv \left(\frac{f\left(c_t^1\left(s^t\right)\right)}{f\left(c_t^2\left(s^t\right)\right)}\right)$$

Moreover, in equilibrium, $q_{t+1}^*\left(s^t\right)$ *is an increasing function of* $\Lambda_t\left(s^t\right)$.

P roof. The first part follows from the fact that equation (1) for the $N = 2$ case implies that

$$\begin{aligned}\frac{v_{t+1}^{1}(q;s^{t})}{1-v_{t+1}^{1}(q;s^{t})} &= \left(\frac{v_{0}^{1}(q)}{1-v_{0}^{1}(q)}\right)\left[\prod_{\tau=0}^{t}\left(\frac{1}{\phi_{\tau}(s^{\tau})}\right)\right] \\ &= \left(\frac{v_{0}^{1}(q)}{1-v_{0}^{1}(q)}\right)\frac{1}{\Lambda_{t}(s^{t})}\end{aligned}$$

where $1/\phi$ was defined in the statement. Solving for $v_{t+1}^{1}(q;s^{t})$ equation (2) follows. The second part follows from the fact that market clearing implies:

$$v_{t+1}^{1*}\left(q_{t+1}^{*};s^{t}\right) = \frac{a_{2}\Lambda_{t}(s^{t})}{a_{2}\Lambda_{t}(s^{t})-a_{1}}$$

Given that $a_1 < 0$ then the right hand side is a strictly increasing function of $\Lambda_t(s^t)$. Since the weight v_{t+1}^{1} is strictly increasing from the proof of proposition 1 then q_{t+1}^{*} is a strictly increasing function of $\Lambda_t(s^t)$. ■

Note that the consumption at the end of period t is defined as follows.

$$c_t(s^t) = \left[\bar{W} + X_t(s^t)\right]^{+}$$

where

$$X_t\left(s^{t-1},s_t\right) = \begin{cases} q_t^{*}\left(s^{t-1}\right),\ s_t = 1,\ a_t = -1 \\ -q_t^{*}\left(s^{t-1}\right),\ s_t = 1,\ a_t = +1 \\ q_t^{*}\left(s^{t-1}\right) - 1,\ s_t = 2,\ a_t = -1 \\ q_t^{*}\left(s^{t-1}\right) + 1,\ s_t = 2,\ a_t = +1 \end{cases}$$

Assume that

$$f\left(c_t^n(s^t)\right) \equiv \exp\left(c_t^n(s^t)\right)$$

Hence, the next result is just a trivial consequence of this process.

Proposition 3 *Suppose that*

$$\Lambda_{t-1}\exp\left[2\left(1-q_t(\Lambda_{t-1})\right)\right]$$

is a bounded function of Λ_{t-1} *for every* t. *Then* $\{q_t^{*}\}_{t=0}^{\infty}$ *is a bounded process and so is* $\{c_t^{*}\}_{t=0}^{\infty}$.

P roof. By definition of Λ_t, we have that

$$\Lambda_t = \Lambda_{t-1}\exp\left[2\left(1-q_t(\Lambda_{t-1})\right)\right]$$

Therefore, $\Lambda_t \leq A$, for some $A > 0$ and all t. Since q_{t+1} is a strictly increasing function of Λ_t then $q_{t+1} \leq B$, for some $B > 0$. Finally note that $c_t(s^t) \leq \bar{W} + 1 + |q_t|$. Therefore $c_t(s^t)$ is also bounded. ■

The idea is to study the long run behavior. Given the results in [5] one hopes to find a suitable definition of long run equilibrium consistent in some way to the standard one. This is because Easley and Rustichini [5] show the convergence of weights to expected utility maximizing preferences given the law of motion in equation (1). Therefore one expects that in this context the long run situation includes not only optimality of actions but also market clearing condition. The following definition has all these features. Let u be an increasing function of c^n.

Definition 2 *Given $u(c^n(s))$ a lottery long run equilibrium for this economy is a price $\bar{q}$ in $[0,\infty)$ for the asset and a lottery $\bar{\mu}$ in $\Delta(A^*)$ such that:*

1. *Given $\bar{q}$, then the lottery $\bar{\mu}$ solves*

$$\max_{\mu \in \Delta(A^*)} \sum_{n=1}^{2} \mu_n \left[\sigma u(c^n(1)) + (1-\sigma) u(c^n(2))\right]$$

s.t.

$$\bar{q}(\mu_1 - \mu_2) \leq \bar{W}$$

and

$$\begin{aligned} c^n(s) &= \bar{W} - \bar{q}a^n + (s-1)a^n \\ a^1 &= -1; \quad a^2 = +1 \end{aligned}$$

2. *The price $\bar{q}$ clears the asset market:*

$$\bar{\mu}_1(\bar{q}) = \bar{\mu}_2(\bar{q})$$

Note then that in this equilibrium $\bar{\mu}_1(\bar{q}) = 0.5$. But then the in this lottery long run equilibrium the budget constraint is not binding. The remaining task is to find a utility function such that $\left(\frac{1}{2}; \frac{1}{2}\right)$ is the solution of the optimization problem. If this is so we would obtain the equilibrium. However the only family of utility functions that matches this definition is $u(c) = U$ for all c. For the first order condition of the maximization problem stated in the definition is

$$\sigma u\left(c^1(1)\right) + (1-\sigma) u\left(c^1(2)\right) = \sigma u\left(c^2(1)\right) + (1-\sigma) u\left(c^2(2)\right)$$

Note that $c^1(2) = c^2(2)$. Since $\sigma > 0$ then $u(c^1(1)) = u(c^2(1))$. Because $\bar{c}^1(1) = \bar{W} + 1 > \bar{W} - 1 = c^2(1)$, then the only solution to this functional equation is the constant function.

This shows that with two actions the limit fails to capture the monotonicity properties of standard utility functions. In other words, the last economy is stationary but the behavior of agents is not really consistent with the standard *monotone* preferences. This is an undesirable property since in [5] the utility functions are indeed strictly increasing. Hence two action economies do not seem to be interesting enough to analyze long run dynamics.

The following section presents examples of three action economies. Although uniqueness is not ensured in this case, unique solutions of the market clearing condition are obtained. Then I also analyze the dynamics of prices and consumption in this cases.

4 Examples of three-action economies.

Let $a^1 = -1$, $a^2 = 0$ and $a^3 = 1$. This means that the investor can buy or sell short exactly one unit of the security, or do nothing. I present two different cases together with the analysis of the dynamics.

4.1 A Continuous Weight example.

Suppose that the date 0 weight on action a^1 is given by:

$$v_0^1(q) = \frac{q}{q + \sqrt{q} + 1}$$

$$v_0^2(q) = \frac{\sqrt{q}}{q + \sqrt{q} + 1}$$

so that

$$v_0^3(q) = \frac{1}{q + \sqrt{q} + 1}$$

It is obvious that these weights satisfy assumption 2. Then date 0 market clearing implies

$$q_0^* = 1$$

Then in period 0 the equilibrium weights are $v_0^{n*} = 1/3$.

Consider the following transition function. Assume that $f\left(c_t^n\left(s^t\right)\right) \equiv \exp\left[\omega\left(c_t^n\left(s^t\right)\right)\right]$, with $\omega > 1$. This satisfies all assumptions stated in section 2. Recall that

$$\frac{1}{\phi_t\left(s^t\right)} \equiv \frac{f\left(c_t^1\left(s^t\right)\right)}{f\left(c_t^2\left(s^t\right)\right)}$$

In our case this implies

$$\begin{aligned}\frac{v_{t+1}^1\left(s^t\right)}{v_{t+1}^2\left(s^t\right)} &= \frac{\sqrt{q}}{\Lambda_t\left(s^t\right)}\\ \frac{v_{t+1}^1\left(s^t\right)}{v_{t+1}^3\left(s^t\right)} &= \frac{q}{\Lambda_t\left(s^t\right)}\end{aligned}$$

This is easily shown by induction. Then the equilibrium weights are

$$\begin{aligned}v_{t+1}^1\left(s^t\right) &= \frac{q}{q+\Lambda_t\left(s^t\right)\sqrt{q}+\Lambda_t\left(s^t\right)}\\ v_{t+1}^2\left(s^t\right) &= \frac{\Lambda_t\left(s^t\right)\sqrt{q}}{q+\Lambda_t\left(s^t\right)\sqrt{q}+\Lambda_t\left(s^t\right)}\\ v_{t+1}^3\left(s^t\right) &= \frac{\Lambda_t\left(s^t\right)}{q+\Lambda_t\left(s^t\right)\sqrt{q}+\Lambda_t\left(s^t\right)}\end{aligned}$$

and then the equilibrium price is

$$q_{t+1}^*\left(s^t\right) = \Lambda_t\left(s^t\right)$$

This gives the equilibrium values for weights:

v_{t+1}^{1*}	v_{t+1}^{2*}	v_{t+1}^{3*}
$\frac{1}{2+\sqrt{\Lambda_t(s^t)}}$	$\frac{\sqrt{\Lambda_t(s^t)}}{2+\sqrt{\Lambda_t(s^t)}}$	$\frac{1}{2+\sqrt{\Lambda_t(s^t)}}$

for all t. Then:

$$q_{t+1}^*\left(s^{t-1}, s_t\right) = q_t^*\left(s^{t-1}\right)\phi_t^*\left(s^{t-1}, s_t\right)$$

The next result establish an upper bound for $\Lambda_t\left(s^t\right)$ (the obvious lower bound is 0).

Proposition 4 *Suppose* $f\left(c_t^n\left(s^t\right)\right)=exp\left[\omega\left(c_t^n\left(s^t\right)\right)\right]$. *The process* $\Lambda_t\left(s^t\right)$ *is uniformly bounded by* $e^{2\omega-1}/2\omega$.

P roof. Since $f\left(c_t^n\left(s^t\right)\right) = \exp\omega\left(c_t^n\left(s^t\right)\right)$, then*to*
$\phi_t\left(s^{t-1},1\right) = \exp\left[-2\omega\Lambda_{t-1}\left(s^{t-1}\right)\right]$
$\phi_t\left(s^{t-1},2\right) = \exp\left[2\omega\left(1-\Lambda_{t-1}\left(s^{t-1}\right)\right)\right] withto$ $\phi_0(1) = \exp\left[-2\omega\right]$
$\phi_0(2) = 1$Therefore*to* $\Lambda_t\left(s^t\right) = \prod_{\tau=0}^{t}\phi_\tau\left(s^\tau\right) = \Lambda_{t-1}\left(s^{t-1}\right)\phi_t\left(s^{t-1},s_t\right)$
$< \Lambda_{t-1}\left(s^{t-1}\right)\exp\left[2\omega\left(1-\Lambda_{t-1}\left(s^{t-1}\right)\right)\right] Now, therighthandside, asafunctionof \Lambda_{t-1}\left(s^{t-1}\right)$
is bounded above. Indeed the function $\exp\left[2\omega\left(1-x\right)\right]$ has a global maximum on $[0,\infty)$ at $x^* = 0.5$, attaining the value $e^{2\omega-1}/2\omega$. ThereforeSince t is arbitrary $\Lambda_t(s^t)$ is bounded. ■

The main implication is that $E\left(q_t\right) \leq e^{2\omega-1}/2\omega$.On the other hand note that from the proof of the last proposition:

$$\begin{aligned} E_{t-1}\left(q_{t+1}\right) &= \Lambda_{t-1}\left[\sigma\exp\left(-2\omega\Lambda_{t-1}\left(s^{t-1}\right)\right)\right. \\ &\quad \left.+\left(1-\sigma\right)\exp\left(2\omega\left[1-\Lambda_{t-1}\left(s^{t-1}\right)\right]\right)\right] \end{aligned}$$

Note that this cannot be characterized as a submartingale or a supermantingale. This is not surprising given the assumption of bounded rationality. Because agents just choose according to the preference and evolution rules given above (maybe due to some type of unawareness in the sense of [9]). Then it is clear that the martingale property is not present here. This is still empirically plausible, as recent empirical contributions emphasized[4].

Note that the dynamical equation $\Lambda_t = \Lambda_{t-1}\exp\left(-2\omega\Lambda_{t-1}\right)$ converges to 0 for any initial condition. The difference equation $\Lambda_t = \Lambda_{t-1}\exp\left(2\omega\left[1-\Lambda_{t-1}\right]\right)$ instead has two fixed points. The origin is not locally stable (it can be shown that the linear Taylor approximation around 0 has a coefficient larger than one). The other fixed point is 1. At this point the eigenvalue is greater than 1. For several $\omega > 1$ this dynamics satisfies the conditions for chaos properties to hold (see theorems 7.2 and 7.3 in [3]). It can be shown from the bifurcation diagram that for values of ω between 1.9 and 2 for example, the behavior of this equation is clearly chaotic.

BIFURCATION DIAGRAM TO BE ADDED.

For values of $\omega > 1$ such that this chaotic behavior holds, then the price equilibrium process is ergodic. The obvious ergodic set is given by $\left[0, e^{2\omega-1}/2\omega\right]$. In fact one can note that each realized history $\left\{s^t\right\}_{t=0}^{\infty}$ determines the realized value of the process $\left\{\Lambda_t\right\}_{t=0}^{\infty}$. In other words, there is an bijection between histories $\left\{s^t\right\}_{t=0}^{\infty}$ and realizations $\left\{\Lambda_t\right\}_{t=0}^{\infty}$. Then this should imply that the property

$$\frac{1}{T+1}\sum_{k=0}^{T}\Pr\left[\theta^k A\cap B\right] = \Pr\left(A\right)\Pr\left(B\right)$$

[4]See for example the literature on the rejection of the Random Walk Hypothesis in [7].

for A and B in the sigma field generated by realizations of $\{\Lambda_t\}_{t=0}^{\infty}$. By Theorem 13.13 in [2] the process is ergodic.

The consumption process is:

$$c_t\left(s^t\right) = \bar{W} + X_t\left(s^t\right)$$

where

$$X_t\left(s^{t-1}, s_t\right) = \begin{cases} \Lambda_{t-1}\left(s^{t-1}\right), & \text{with probability } \frac{\sigma}{2} \\ -\Lambda_{t-1}\left(s^{t-1}\right), & \text{with probability } \frac{\sigma}{2} \\ \Lambda_{t-1}\left(s^{t-1}\right) - 1, & \text{with probability } \frac{1-\sigma}{2} \\ -\Lambda_{t-1}\left(s^{t-1}\right) + 1, & \text{with probability } \frac{1-\sigma}{2} \end{cases}$$

and where $\bar{W}>0$.Clearly $c_t\left(s^t\right)$depends entirely on the dynamics of Λ_t. Since Λ_t is ergodic, so is c_t. In fact the vector $\left\{\left\{c_t^n\left(s_t, q_t\left(s^{t-1}\right)\right)\right\}_{n=1}^2\right\}_{s_t=1}^2$ defined as

$$c^n\left(s_t, q_t\left(s^{t-1}\right)\right) = \bar{W} + a^n q\left(s^{t-1}\right) + \left(s_t - 1\right) a^n$$

is ergodic and then the vector converges weakly. The main point now is to see whether the limit can be interpreted as an equilibrium in the sense of definition 2. To do this, note first that the equilibrium weights also are ergodic. On average those weights should converge by the Birkhoff Ergodic Theorem to the mean weight with respect to the invariant ergodic measure (see [4], chapter 6). In fact this same result implies that the process Λ_t converges in distribution to an invariant ergodic measure.

Suppose λ is the ergodic measure. This is defined on the sigma field of subsets in $\Omega \equiv \left[0, e^{2\omega-1}/2\omega\right]$. The question is whether we can interpret this as a long run equilibrium. The problem is that the definition of lottery long run equilibrium includes as states of nature only the asset shocks. What we do is to enlarge the state space to the whole ergodic set in order to have a sensible long run stationary equilibrium.

Definition 3 *The stationary long run equilibrium for a given (measurable) function $u(c)$ for the economy in subsection 3.1 is a random vector q defined on $\left[0, e^{2\omega-1}/2\omega\right]$ with distribution λ, a random vector $\bar{c}$ and a vector of lotteries μ in $\Delta\left(A^*\right)$ that satisfies*

1. *Given $\bar{q}$, for each w then $\bar{\mu}$ solves*

$$\max \sum_{n=1}^{3} \mu^n \left[\sigma u\left(c^n(1, q)\right) + (1-\sigma) u\left(c^n(2, q)\right)\right]$$

subject to

$$\begin{aligned} c^n(s,q) &= \bar{W} + a^n q + (s-1)a^n \\ a^1 &= 1; a^2 = 0; a^3 = -1 \end{aligned}$$

2. The lottery satisfies market clearing.

$$\mu^1(q) = \mu^3(q)$$

λ-a.s.

This definition includes the fact that q is itself a random variable whose realization is first known to the agent. This is realized according to the ergodic measure λ defined on the domain of q. Note that in the long run the limit of weights should also be dependent on q. What we would like is to have that

$$\mu^n = \lim_{t\to\infty} v_t^{n*}$$

for some utility function $u(c)$. Then we would like to have

$$\begin{aligned} \mu^1(q) &= \mu^3(q) = \frac{1}{2+\sqrt{q}} > 0 \\ \mu^2(q) &= \frac{\sqrt{q}}{2+\sqrt{q}} \end{aligned}$$

where q is in the ergodic set. Since $\lambda(0) = 0$ (that is, the probability under the ergodic measure that in the long run $q = 0$ is also zero) then $\mu^2(q) > 0$ $\lambda - a.s.$ But then from the first order conditions of the optimization problem given in the last definition we need to have that

$$\begin{aligned} \left[\sigma u\left(c^1(1,q)\right) + (1-\sigma)u\left(c^1(2,q)\right)\right] &= \left[\sigma u\left(c^3(1,q)\right) + (1-\sigma)u\left(c^3(2,q)\right)\right] \\ &= \left[\sigma u\left(c^2(1,q)\right) + (1-\sigma)u\left(c^2(2,q)\right)\right] \end{aligned}$$

Now, clearly we know that $c^2(s,q) = \bar{W}$, and then $[\sigma u(c^2(1,q)) + (1-\sigma)u(c^2(2,q))] = u(\bar{W})$. On the other hand, clearly $c^1(1,q) > \bar{W} > c^3(1,q)$. Therefore, to get the equality above it is necessary to have

$$c^1(2,q) < W < c^3(2,q)$$

which implies

$$q < 1$$

However $\lambda[0,1] < 1$. Hence we can say that the economy does not converge almost surely to a stationary long run equilibrium (even if u are state dependent). In some sense, this economy will in the long run be in the stationary equilibrium with probability $\lambda[0,1]$. Still this is somehow better than the two action case, in which it was never possible to get a long run equilibrium with strictly increasing u functions.

4.2 Piecewise Linear Weights

Suppose that:

$$v_0^1 = \begin{cases} \alpha q, & 0 \le q \le \frac{1}{\alpha} \\ 1, & q > \frac{1}{\alpha} \end{cases}$$

and

$$v_0^2 = \begin{cases} \alpha q, & 0 \le q \le \frac{1}{2\alpha} \\ 1 - \alpha q, & \frac{1}{2\alpha} \le q \le \frac{1}{\alpha} \\ 1, & q > \frac{1}{\alpha} \end{cases}$$

with $0 < \alpha < 1$. Then $v_0^3 = 1 - v_0^1 - v_0^2$. The date 0 equilibrium is given by

$$q_0^* = \frac{1}{3\alpha}$$

which is clearly less than $1/2\alpha$. The date 0 equilibrium weights are again $v_0^{n*} = 1/3$ for $n = 1, 2, 3$. Given the law of motion defined before, we get that for $0 \le q \le 1/2\alpha$

$$\frac{v_{t+1}^1}{v_{t+1}^2} = \left(\frac{1}{\Lambda_t(s^t)}\right)$$

and

$$\frac{v_{t+1}^1}{v_{t+1}^3} = \left(\frac{\alpha q}{1 - 2\alpha q}\right)\left(\frac{1}{\Lambda_t(s^t)}\right)$$

The proof is again by induction. In a similar way, for prices $1/2\alpha < q \le 1/\alpha$, the transitions are given by the following.

$$\frac{v_{t+1}^1}{v_{t+1}^2} = \left(\frac{\alpha q}{1 - \alpha q}\right)\left(\frac{1}{\Lambda_t(s^t)}\right)$$

where $v^3_{t+1}(q) = 0$ for this domain. This gives the following weights at period t.

$$v^1_{t+1}\left(s^t\right) = \begin{cases} \frac{\alpha q}{\alpha q(1+\Lambda_t(s^t))+\Lambda_t(s^t)(1-2\alpha q)}, & 0 \le q \le \frac{1}{2\alpha} \\ \frac{\alpha q}{\alpha q+(1-\alpha q)\Lambda_t(s^t)}, & \frac{1}{2\alpha} < q \le \frac{1}{\alpha} \\ 1, \quad q > \frac{1}{\alpha} \end{cases}$$

$$v^2_{t+1}\left(s^t\right) = \begin{cases} \frac{\alpha q \Lambda_t\left(s^t\right)}{\alpha q(1+\Lambda_t(s^t))+\Lambda_t(s^t)(1-2\alpha q)}, & 0 \le q \le \frac{1}{2\alpha} \\ \frac{(1-\alpha q)\Lambda_t\left(s^t\right)}{\alpha q+(1-\alpha q)\Lambda_t(s^t)}, & \frac{1}{2\alpha} < q \le \frac{1}{\alpha} \\ 0, \quad q > \frac{1}{\alpha} \end{cases}$$

and then the equilibrium asset price is

$$q^*_{t+1} = \frac{\Lambda_t\left(s^t\right)}{\alpha\left(1 + 2\Lambda_t\left(s^t\right)\right)}$$

which is always in $[0, 1/2\alpha)$. The equilibrium weights are

v^{1*}_{t+1}	v^{2*}_{t+1}	v^{3*}_{t+1}
$\frac{1}{\Lambda_t(s^t)+2}$	$\frac{\Lambda_t\left(s^t\right)}{\Lambda_t(s^t)+2}$	$\frac{1}{\Lambda_t(s^t)+2}$

The consumption allocation in the BR equilibrium is as follows.

$$c_t\left(s^t\right) = \begin{cases} \bar{W} + \frac{\Lambda_t\left(s^t\right)}{\alpha(1+2\Lambda_t(s^t))}, & \text{with probability } \frac{\sigma}{2} \\ \bar{W} - \frac{\Lambda_t\left(s^t\right)}{\alpha(1+2\Lambda_t(s^t))}, & \text{with probability } \frac{\sigma}{2} \\ \bar{W} + \frac{\Lambda_t\left(s^t\right)}{\alpha(1+2\Lambda_t(s^t))} - 1, & \text{with probability } \frac{1-\sigma}{2} \\ \bar{W} - \frac{\Lambda_t\left(s^t\right)}{\alpha(1+2\Lambda_t(s^t))} + 1, & \text{with probability } \frac{1-\sigma}{2} \end{cases}$$

$$\bar{W} > B > 0$$

where B is to be determined. The evolution of the equilibrium price depends again entirely on Λ. This process follows the following law of motion:

$$\Lambda_t\left(s^{t-1}; s_t\right) = \Lambda_{t-1}\left(s^{t-1}\right)\phi_t\left(s^{t-1}; s_t\right)$$

In this case, using the same function f :

$$\phi_t\left(s^{t-1}; 1\right) = \exp\left[-2\omega\left(\frac{\Lambda_{t-1}\left(s^{t-1}\right)}{\alpha\left(1 + 2\Lambda_{t-1}\left(s^{t-1}\right)\right)}\right)\right]$$

$$\phi_t\left(s^{t-1};2\right)=\exp\left[2\omega\left(1-\left(\frac{\Lambda_{t-1}\left(s^{t-1}\right)}{\alpha\left(1+2\Lambda_{t-1}\left(s^{t-1}\right)\right)}\right)\right)\right]$$

Since $\Lambda_{t-1}\left(s^{t-1}\right)\geq 0$, $\omega>0$ and $\alpha\in(0,1)$ clearly $\Lambda_t\left(s^{t-1};s_t\right)\leq\Lambda_{t-1}\left(s^{t-1}\right)$ $\exp\left[2\left(1-\left(\frac{\Lambda_{t-1}\left(s^{t-1}\right)}{\alpha(1+2\Lambda_{t-1}(s^{t-1}))}\right)\right)\right]$. Suppose now that $\alpha<\omega/2$. Then the function

$$y=x\exp\left[2\left(1-\left(\frac{x}{\alpha\left(1+2x\right)}\right)\right)\right]$$

attains a global maximum at the value

$$x^*=\frac{\left(1-2\left(\frac{\alpha}{\omega}\right)\right)+\sqrt{1-2\left(\frac{\alpha}{\omega}\right)}}{4\left(\frac{\alpha}{\omega}\right)}$$

Define $B\equiv x^*\exp\left[2\omega\left(1-\left(\frac{x^*}{\alpha(1+2x^*)}\right)\right)\right]$. Then with $\alpha<\omega/2$ the price process is again bounded and so the consumption process is strictly positive with $\bar{W}>B$. The process given by $\Lambda_t\left(s^{t-1};s_t\right)=\Lambda_{t-1}\left(s^{t-1}\right)$ $\exp\left[-2\omega\left(\frac{\Lambda_{t-1}\left(s^{t-1}\right)}{\alpha(1+2\Lambda_{t-1}(s^{t-1}))}\right)\right]$ has a unique stable steady state given by the origin. On the other hand, the equation $\Lambda_t\left(s^{t-1};s_t\right)=\Lambda_{t-1}\left(s^{t-1}\right)\exp\left[2\omega\left(1-\left(\frac{\Lambda_{t-1}\left(s^{t-1}\right)}{\alpha(1+2\Lambda_{t-1}(s^{t-1}))}\right)\right)\right]$ has an unstable trivial stationary point (the origin) and a stable positive steady state[5], given by

$$\bar{\Lambda}^2=\frac{\alpha}{1-2\alpha}$$

for any (ω,α) that satisfies $\alpha<\omega/2$. Again by the same argument as before the whole process $\Lambda_t\left(s^t\right)$ is ergodic. The ergodic set is $[0,B]$. As before, due to the first order conditions of the optimization problem in the stationary long run equilibrium, it can be shown again that there is an equilibrium as long as $q<1$. However in this case there are several values for ω and α such that B is less than one. For example, if $\omega=3/2$ and $\alpha=0.25$, then $B\approx 0.6747$. Then for (possible $s-$dependent) utility functions u the economy converges to a stationary long run equilibrium. For example, if $u_1(c)=c$ is the state 1 utility function and $u_2=u_1-1$ is the state 2 utility function then the equilibrium conditions are automatically satisfied.

[5]The derivative of the function $y=x\exp\left[2\omega\left(1-\frac{x}{\alpha(1+2x)}\right)\right]$ is $y'=\exp\left[2\omega\left(1-\frac{x}{\alpha(1+2x)}\right)\right]\left(1-\frac{2\omega x}{\alpha(1+2x)^2}\right)$. At the positive steady state, $\exp\left[2\omega\left(1-\frac{x}{\alpha(1+2x)}\right)\right]=1$. Then the value of this derivative is given by $\left(1-\frac{2\omega x}{\alpha(1+2x)^2}\right)$ evaluated at $\bar{x}=\bar{\Lambda}^2$. It can be shown that at this value the bracket is negative but greater than -1. Then the positive steady state is stable.

5 Concluding remarks

The last two examples show that, although the same type of replicator dynamics as in [5] are used, the boundedly rational equilibrium process does not necessarily converge in distribution to the stationary long run equilibrium. An interesting interpretation of this equilibrium concept is its relation with the sunspot idea. What happens there is that nature chooses q according to the ergodic distribution λ. After this traders choose the optimal lottery that maximized expected utility. The relation comes from the fact that all agents coordinate in the same value q. This is not related to the realization of s, as the definition states. Then, this can be interpreted as some sort of extrinsic uncertainty that is realized before agents take decisions. However the stationary long run equilibrium is not a special case of a sunspot equilibrium since the state s is not extrinsic.

As a consequence of this, several directions can be taken. First it is necessary to refine the dynamics analysis to more actions and states, in order to generalize the examples. It is also convenient to see whether there are other types of dynamics with different transitions on preferences. A closer look at the stationary long run equilibrium needs to be done, in order to explore further it relationship with the traditional competitive equilibrium concept. This includes a better analysis of long run equilibrium in its connection with sunspot equilibria.

Finally, as mentioned in the introduction, more complex assets need to be considered in this context. Specially important seem to be long term securities. This usually would imply more complex decisions at each date, that includes also consumption decisions (in my paper consumption is just the residual after portfolios are decided). The most problematic point is, to my view, the issue of how wealth would evolve through time. This should include bankruptcies issues, that make the period t equilibrium harder to define and compute. Still I find this the most exciting extension of this type of asset market application of the replicator dynamics framework.

References

[1] Cass, David and Karl Shell (1983). "Do Sunspot Matter?", *Journal of Political Economy*, 91, 193-227.

[2] Davidson, James (1994). *Stochastic Limit Theory.* Oxford, Oxford University Press.

[3] Day, Richard (1994). *Complex Economic Dynamics, Volume 1.* Cambridge, MA, MIT Press.

[4] Durret, Richard (1996). *Probability: Theory and Examples (2nd edition).* California, Wadsworth Publishing Company.

[5] Easley, David and Aldo Rustichini (1999) "Choice without Beliefs", *Econometrica*, 67, 1157-1184.

[6] Fudenberg, Drew and David Levine (1998). *Theory of Learning in Games.* Cambridge, MA, MIT Press.

[7] Lo, Andrew and Craig MacKinlay (1999). *A Non-Random Walk Down Wall Street.* Princeton, NJ, Princeton University Press.

[8] Modica, Salvatore and Aldo Rustichni (1994) "Awareness and Partitional Information Structures", *Theory and Decision*, 37, 107-124.

[9] Modica, Salvatore and Aldo Rustichni (1999) "Unawareness and Partitional Information Structures", *Games and Economic Behavior*, 27, 265-298.

[10] Rubinstein, Ariel.(1998) *Modelling Bounded Rationality.* Cambridge, MA, MIT Press.

[11] Sargent, Thomas (1993). *Bounded Rationality in Macroeconomics.* Cambridge University Press.