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General Equilibrium with Unawareness in Economies with Production.

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Abstract

This paper investigates two types of GEI models with production and unawareness, extending the framerwork of Modica et al (*Economic Theory*, 1998). Existence of equilibrium in an entrepreneurship economy is guaranteed given their same assumptions coped with decreasing-returns-to-scale technologies. A non-existence example is provided to show the necessity of some minimum awareness by agents. Two results from the standard GEI with production framework still hold: the maximum net present value characterization of equilibrium output and the Modigliani-Miller Theorem. A modified risk-evaluation-by-owners formula is also provided. An example of creation of a new good shows that it is not always the case that a non-trivial change in awareness leads to a different equilibrium outcome. In an ownership economy, the Pareto ownership equilibrium is characterized. As a side result, a modified version of the Minkowski-Farkas lemma is proved. Finally, it is shown that unanimity in production decisions still holds under multiplicative uncertainty.

1 Introduction

General Equilibrium Theory with Incomplete Markets (*GEI*) has already a long history. Its development was not only important in a purely methodological sense but also for applications in asset pricing and macroeconomics. The modelling of uncertainty and risk in this framework was based mainly

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on the well-known Savage's [13] decision theory. This has been standard for many years.

This theory has still been subject to major criticisms. For example, important objections have come from Experimental Economics, as surveyed, e.g., in Rubinstein [12]. More importantly, the Savage paradigm intrinsically assumes that the decision process takes into account all possible future contingencies. This assumption could be suitable for simple problems, in which the number of events to be considered is small. However, most of the typical non-trivial economic issues involve *too many* contingencies to be aware of. For instance, it would be arguable to assume that an investor is always aware of all possible facts that could influence her decision of investing in Internet stocks. Nevertheless the standard GEI literature, as well as applications in macroeconomics and finance, kept using Savage's theory, given the lack of alternatives.

Recently a new literature on alternative foundations of decision theory under uncertainty has been developed. One of the contributions has been the work by Modica and Rustichini (see [7] and [8]). These two authors developed a formalization of the concept of *unawareness*, as well as its axiomatic characterization. This is done mainly through a logical system with limited reasoning ability of the subject. This is important since it constitutes the first foundation for decision theory without awareness of all contingencies. In Modica et. al. [9], a first application of the concept of unawareness in the general equilibrium tradition is found. In brief, the main difference with the standard approach is that, with unawareness, agents must re-optimize in the future states that agents did not consider in the first period. This gives different implications in terms of conditions of existence. Still some classical results in the GEI literature such as nominal indeterminacy arises in the presence of unawareness.

This paper considers a first extension of [9] to production economies. The motivation for this extension, to my view, is not trivial. In a provocative paper, Arrow and Hahn [1] have recently emphasized the role of uncertainty modelling in traditional General Equilibrium Theory. One conclusion of that paper is the convenience of finding an alternative way to introduce uncertainty. As mentioned, [9] constitutes the first attempt to analyze unforeseen contingencies in the competitive equilibrium model. A pure exchange economy, though, is not suitable to capture how unawareness influences production decisions. This is essential when inventions and innovations are taken into account. Arrow and Hahn [1] emphasize the fact that hardly agents can be aware of all possible events when considering innovations. Therefore the equilibrium investment and consequent output levels could change dramatically among economies with different degrees of unawareness.

The first case considered in the paper is the simplest one, i.e., the entrepreneurship economy with one physical commodity. Here consumers own firms (production technologies). They decide production quantities as well as the portfolio of assets. Under the assumptions on preferences and states in [9], and assuming decreasing returns to scale in production, existence of equilibrium is guaranteed. This may not be too surprising since the ownership structure is the simplest possible. However, assumption C in [9] cannot be eliminated, as it is shown through example (modified from [9]). The interpretation is that all consumers must be aware of at least the states spanned by the asset structure. This is clearly a serious limitation of the model. However dropping this assumption is out of the scope of this paper, and so it is maintained.

Under entrepreneurship many of the classical results in GEI with production still hold, provided that some small changes in definitions are introduced. The equilibrium production allocation can be characterized by the solution to a *modified net present value* maximization problem. The net present value formula includes an unawareness modification. That is, the expected net present value uses the martingale prices over the states foreseen ex-ante by the owner. This emphasizes the role of unawareness in production. In the extreme, it could happen that the owner is not aware of the productivity of its firm and decides to invest zero in her own technology.

In the entrepreneurship economy a modified version of the Modigliani-Miller theorem holds. For the firm's owner the financial decisions can be taken independently of her production decisions. All that matters is the combined production-consumption decision, as in the standard literature. The main reason for the MM theorem to hold here is the ownership structure. Since firms are owned by individual agents (with full responsibility in foreseen contingencies), and since bankruptcy only occurs in unforeseen states, then it should be obvious that the financial decisions and output decisions are independent. The lesson here is that unawareness does not matter for the indeterminacy of financial structure in entrepreneurship economies.

In the same context a characterization of risk evaluated by entrepreneurs is presented. Although the main formula is similar to the standard [1] I find one difference. With the riskless asset the net present value of the equilibrium output allocation is equal to the expected NPV discounted at the riskless rate plus the covariance between state prices and output levels in the standard framework. This formula breaks down here due to unawareness. Unforeseen contingencies implies possible bankruptcy. This makes a bond with unit constant payoff become a risky asset. This must be taken into account and

[1] See [5] for the standard formula.

the mentioned standard formula does not hold here.

I also provide an example with two agents, where one entrepreneur owns a technology able to create a new physical commodity in one state. The main purpose of this exercise is to see whether is always the case that a change in unawareness in one agent leads to different production allocations. The answer from this example is: not necessarily. I consider the situation when the agent who lends to the entrepreneur is not aware (ex-ante) of the possibility of a new good in the future. Then I switch to the case in which both agents are aware of the invention. It turns out that the equilibrium in the latter economy is identical to the one in the first case considered. Hence this example questions the potential dependence of the production allocation on the awareness of agents.

I finally present a first attempt to treat ownership economies. I introduce a modified definition of Pareto Partnership equilibrium. I characterize the equilibrium production decision as the solution to the standard NPV maximization problem. As a side result, a modified version of the Minkowski-Farkas lemma is shown. I also show that Diamond's [2] multiplicative uncertainty assumption in this context still gives unanimity in output allocation. Hence these two results suggest that unawareness does not change the main characterization propositions in the ownership model.

These results lead to the following conclusion. Economies with simple ownership structures and incomplete markets do not seem to change dramatically with unawareness. Most of the classical results still hold with slight modifications. However, real life firms usually take the form of corporations, ignored in this paper. I discuss the possible issues of this extension in the conclusions. Another lesson can be gotten here. The degree of unawareness may or may not alter the equilibrium. The mentioned example clarifies this point, although more work is needed to confirm this in a generic way.

The paper is organized as follows. Section 2 presents the entrepreneurship economy, and it states the assumptions to ensure existence. It also contains an example, modified from [9] introducing production, to show non-existence. Section 3 includes three main characterization results from the GEI literature applied to the unawareness case. It shows that the classical NPV characterization still holds, as well as the MM theorem. It also has the risk evaluation formula by the owners. Section 4 presents the example of creation of a new good in the entrepreneurship context. Section 5 introduces the partnership economy. It includes the NPV maximization characterization proposition of equilibrium output, and the unanimity result, modified from Diamond [2]. Section 6 presents the main issues not treated here, as well as some other possible extensions and some concluding remarks.

2 The Entrepreneurship Economy

2.1 The Environment

Consider a production entrepreneurial version of Modica et al. [9]. The economy lasts two periods, labeled $t = 0, 1$. In period 1 there is no uncertainty. At date 1, S possible states can occur, with $S < \infty$. Assume for now a unique physical commodity in each period and state. There is a large number of agents, labeled $i = 1, 2, ..., I$. There is a set of assets, called bonds. They are traded at some price q_j in period 0 and gives a payoff $R_j(s)$ in state s at date 1. There is a set of firms, labeled $k = 1, ..., K$. I assume $K \leq I$. This is so since I consider the simplest production economy version, which is the entrepreneurship economy. Hence for each firm k there is a single agent i that owns that firm. Hence there is a map from k to i. However, to simplify the analysis, I will consider the *inverse* map (from k to i). Here $k(i)$ means the firm k owned by i.

As in the referred paper, I assume that agents cannot be aware of all states. Indeed, each agent i is aware (at date 0) of a subset of states, $S^i \subset S$. Hence the ex-ante utility function is defined over consumption on states S^i, in addition to period 0 consumption. Denote $x^i(s)$ the consumption by i in state s at date 1. Similarly denote x_0^i the consumption by i at date 0. Agent i' s utility at date 0 is

$$u^i\left(x_0^i; \left(x^i(s)\right)_{s \in S^i}\right)$$

However, in period 1 there are states that agent i was not aware of at date 0. For these states we assume a utility function

$$u_s^i\left(x^i(s)\right) \quad s \notin S^i$$

Each agent has a positive endowment $\omega_0^i > 0$ in period 0 and also at date 1, that is, $\omega_1^i(s) > 0$.

Agents $i(k)$ own a firm, given by a technology. For each firm k assume the existence of a production function. This transformation technology produces at period 1 with resources invested at date 0. Given an amount of *investment* in period 0, denoted by y_0^k, the technology gives a level of output equal to $y_1^k(s) = f_s^k\left(y_0^k\right)$[2]. All the firms's owners, called *entrepreneurs*, are assumed to fund their own investments with their endowments and bond trades. For the moment I exclude the possibility of equity issues, although

[2] This is a simple version of more general types of technologies. However, I chose this one since it captures the unawareness part. For, in the case of a technology set, Y^k, given by a transformation function $T^k\left(y^k\right)$,

this constitutes an essential extension of this model. For those agents who are not entrepreneurs assume $\mathbf{y}^i = 0$.

Agents face two types of budget constraints. At date 0 they confront the following budget set.

$$B^i = \left\{ \begin{array}{c} x^i \in \Re^{S^i+1} : x_0^i \leq \omega_0^i + \left(\psi^i - \phi^i\right) \cdot q - y_0^{k(i)} \\ x_1^i(s) \leq \omega_1^i(s) + (R \otimes \kappa)_s \cdot \phi^i - R_s \cdot \psi^i + y_1^{k(i)}(s) \\ \phi^i \cdot \psi^i = 0 \end{array} \right\}$$

Here ϕ^i denotes the amount of bonds bought by agent i at date 0. ψ^i denotes asset sales. The last constraint states that agent i cannot be on both sides of the bond markets. Since agents are not aware of all possible contingencies, it is possible that at some state some agents who sold short assets must go bankrupt. Hence perfect repayment of bonds is not guaranteed. This is why the term $(R \otimes \kappa)_s \cdot \phi^i$ shows up in the constraint. Following [9], the term κ is defined as follows. Let $K_j(s)$ be the repayment rate of bond j for its holders. Define:

$$M^i(s) \equiv \min \left\{ R_s \psi^i ; \omega_1^i(s) + \sum_{j=1}^{J} \left(R_j(s) \times K_j(s)\right) \phi_j^i + y_1^{k(i)}(s) \right\}$$

Let also

$$\eta(i,s) \equiv \left\{ \begin{array}{cc} \frac{M^i(s)}{R_s \cdot \psi^i} & if \quad R_s \cdot \psi^i > 0 \\ 1 & otherwise \end{array} \right.$$

Finally, define

$$\kappa_j(s) = \left\{ \begin{array}{cc} \frac{\sum_{i=1}^{I} \eta(i,s) R_j(s) \psi_j^i}{\sum_{i=1}^{I} R_j(s) \psi_j^i} & \text{if} \quad \sum_{i=1}^{I} R_j(s) \psi_j^i > 0 \\ [0,1] & \text{otherwise} \end{array} \right.$$

In equilibrium, the repayment rate $K_j(s)$ must coincide with $\kappa_j(s)$. However the last map is in fact a correspondence. Hence the equilibrium repayment rate is a fixed point of this correspondence, which in [9] is called the β correspondence, or the book-keeping correspondence.

If the state realized at date 1 is not in S^i, hence agent i must re-optimize. Her budget constraint in this case is as follows.

$$\begin{array}{rcl} B^i(s) & = & \left\{ x_1^i(s) \in \Re_+ : x_1^i(s) - \omega_1^i(s) \leq (R \otimes \kappa)_s \cdot \phi^i - M_s^i + y_1^{k(i)}(s) \right\} \\ s & \notin & S^i \end{array}$$

This states that agent i consumes her income, which can be 0 if $\omega_1^i(s) + \sum_{j=1}^{J}(R_s^j \times \kappa_s^j)\phi_j^i + y_1^{k(i)}(s) < R_s \cdot \psi^i$. In fact, given the assumption of one commodity, it can be shown that

$$\begin{aligned} x^i(s) &= \max\left\{0, (R \otimes \kappa)_s \cdot \phi^i + y_1^{k(i)}(s) - R_s \cdot \psi^i\right\} \\ s &\notin S^i \end{aligned}$$

Finally, I assume that agent i is only aware of S^i. She is not aware of other agent's characteristics and trades, including preferences and technologies. This is essential since otherwise an agent i could infer more states than those included in S^i if she could observe trades by some other i' with $S^i \subset S^{i'}$.

2.2 Equilibrium.

In equilibrium, agent i chooses $\left(x_0^i, (x_1^i(s))_{s\in S^i}, y_0^{k(i)}, \left(y_1^{k(i)}(s)\right)_{s\in S^i}\right)$ such that it maximizes $u^i\left(x_0^i; (x^i(s))_{s\in S^i}\right)$ at date 0 subject to $\left(x_0^i; (x^i(s))_{s\in S^i}\right) \in B^i$. This problem is denoted as CP^i. For states outside her own state space, she just maximizes $u_s^i(x_1^i(s))$ subject to $(x_1^i(s)) \in B^i(s)$, taking the date 0 production and investment decisions as given. Given that there is only one commodity the latter problem is just trivial: she just consumes what is left.

The definition of equilibrium is a straightforward modification of the Modica et. al [9] definition. The only difference is the presence of production decisions by entrepreneurs.

Definition 1 *A competitive equilibrium for the entrepreneurial economy is a set of prices q, repayment rates κ and allocations $\left(x_0^i, (x_1^i(s))_{s\in S}, \phi, \psi\right)$ such that*

i $\left(x_0^i, (x_1^i(s))_{s\in S^i}, y_0^{k(i)}, \left(y_1^{k(i)}(s)\right)_{s\in S^i}\right)$ *solves CP^i, for $i = 1, 2, ..., I$*

ii The following condition is satisfied

$$\begin{aligned} x^i(s) &= \max\left\{0, (R \otimes \kappa)_s \cdot \phi^i + y_1^{k(i)}(s) - R_s \cdot \psi^i\right\} \\ s &\notin S^i \end{aligned}$$

iii $\sum_{i=1}^{I}(x_1^i(s) - \omega_1^i(s)) = \sum_{i=1}^{I} y_1^{k(i)}(s)$ *for each s in S,* $\sum_{i=1}^{I}(x_0^i - \omega_0^i) = \sum_{i=1}^{I} y_0^{k(i)}$ *and* $\sum_{i=1}^{I}(\phi^i - \psi^i) = 0$.

iv κ is a fixed point of the book keeping correspondence.

Following the original setup, I suppose $J \leq \#S^i \leq S$ for all i. Therefore there are incomplete asset markets by assumption. This is somehow crucial to get existence of equilibrium, as it will be shown in brief. Define C as the set of the first J states. Following identical steps as in Modica et al. [9] I present the existence result.

Proposition 2 *Assume the following properties.*

P The matrix R is strictly positive (all its elements are strictly positive).

U Functions u^i and u^i_s are continuous, strictly increasing and strictly concave for all i and $s \notin S^i$

Y Functions f^i_s are continuous, strictly increasing and strictly concave for all i and s in S. Moreover, assume that there is a value $\bar{y} > 0$ such that for any $y > \bar{y}$ then $\max_{s,i} f^i_s(y) < y$. Also assume that for $y > \bar{y}$, and for any $\lambda_s \in \Re_{++}$ then $y - \sum_{s=1}^{S} \lambda_s f^{k(i)}_s(y)$ is strictly increasing.

W Endowments are strictly positive.

C $C \subseteq S^i$ for all $i = 1, 2, \ldots, I$

FR The submatrix R_C, conformed by the first C rows of R is non-singular.

Then a competitive equilibrium exists.

The proof is in the Appendix. Essentially it is the same existence theorem as in [9]. The only extra ingredient is endogenous production. However this does not entail any special difficulty in the current entrepreneur framework, provided that assumption Y holds. This ensures that budget constraints are still compact-valued, as well as continuous. In the case of a Partnership Economy this is relaxed to include constant returns to scale.

The strongest condition is in fact C, as in the original exchange economy case. This is interpreted as a condition of $sufficient\ awareness$ on behalf of agents. These are aware of the span of the existing asset markets. Otherwise equilibrium may not exist. This proved to be true in the original paper. The question is whether is it still important to keep this in the current situation. We see in the next example (modified from Modica et al [9]) that the answer is affirmative.

Example 3 *Assume two consumers (or a continuum with measure one of two types). Consumer 1 owns a firm. Consumer 2 only trades in asset markets. There are two states of the world in period 1.*

Suppose that none of the two consumers care about date 0 consumption. Moreover consumer 1 is only aware of $S^1 = \{1\}$, while consumer 2 of $S^2 = \{2\}$. Date-0 utility functions for each consumer are as follows.

$$\begin{aligned} U^1 &= \ln\left[1 + x^1(1)\right] \\ U^2 &= \ln\left[1 + x^2(2)\right] \end{aligned}$$

Endowments are as in the original example: $\omega^1 = \omega^2 = (1, 1, 1)$. The production technology for agent 1 is $y_s^1 = (s - 1 + \varepsilon)\sqrt{y_0^1}$, where $s = 1, 2$ and $\varepsilon > 0$ and small enough. Hence the technology is only productive in state 2. There are two assets. The payoff matrix is

$$R = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

with $\alpha \in (0, 1)$.

It can be shown (see [9], page 288) that it cannot be that

$$\kappa = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

in any equilibrium. Otherwise arbitrage opportunities arise. Hence there must be trade. On the other hand, note that in equilibrium

$$\begin{aligned} y_0^1 &= \sum_{j=1}^{2} q_j \left(\psi_j^1 - \phi_j^1\right) + 1 \\ 0 &= \sum_{j=1}^{2} q_j \left(\psi_j^2 - \phi_j^2\right) + 1 \end{aligned}$$

The equilibrium implies that $\sum_{i=1}^{2}\left(\psi_j^i - \phi_j^i\right) = 0$ for each j. Hence, this implies that $y_0^1 = 2$. Suppose for a moment that this is the case. Then the budget constraint for agent 1 becomes $1 = \sum_{j=1}^{2} q_j\left(\psi_j^1 - \theta_j^1\right)$.

Let us consider the first case, in which

$\psi_1^1 = 0$	$\phi_1^1 > 0$
$\psi_2^1 > 0$	$\phi_2^1 = 0$

and where the asset trading for agent 2 is completely symmetric. Because of full repayment for the state in which 1 is aware then $\kappa_2(1) = 1$ in this case. From the first period budget constraint we get that

$$q_2\psi_2^1 - q_1\phi_1^1 = 1 \tag{1}$$

Therefore the problem for agent 1 at date 0 is to maximize $\ln\left[1 + x^1(1)\right]$ *subject to the constraint 1 and* $x^1(1) = 1 + \kappa_1(1)\phi_1^1 - \alpha\psi_2^1 + \varepsilon\sqrt{2}$. *The corresponding FOC implies that*

$$\kappa_1(1) = \frac{q_1}{q_2}\alpha$$

On the other hand, from the definition of the β *correspondence (or the book - keeping map) we have*

$$\begin{aligned}\kappa_1(1) &= \eta(2,1)\\ &= \frac{M^2(1)}{\psi_1^2}\\ &= \frac{1+\alpha\phi_2^2}{\psi_1^2}\end{aligned}$$

From agent 2's budget constraint in period 0, we have that $\frac{\phi_2^2}{\psi_1^2} = \frac{q_1}{q_2} + \frac{1}{q_2\psi_1^2}$. *Hence from the book keeping map:*

$$\begin{aligned}\kappa_1(1) &= \frac{1}{\psi_1^2} + \alpha\frac{q_1}{q_2} + \alpha\frac{1}{q_2\psi_1^2}\\ &= \alpha\frac{q_1}{q_2} + \frac{1}{\psi_1^2}\left(1+\frac{\alpha}{q_2}\right)\end{aligned}$$

Therefore, to make both expressions equal we must have

$$\frac{1}{\psi_1^2}\left(1+\frac{\alpha}{q_2}\right) = 0$$

which implies a negative price for asset 2, incompatible with an equilibrium. Then we cannot have an equilibrium with the trading profile for agent 1 as described above.

The second case is the symmetric one. Assume

$\psi_1^1 \geq 0$	$\phi_1^1 = 0$
$\psi_2^1 = 0$	$\phi_2^1 > 0$

Given this profile, it must be the case now that $\kappa_1(1) = 1$ *and* $\kappa_2(2) = 1$. *Agent 1's problem is to maximize the same ex-ante utility function subject to the same definition for* $x^1(1)$ *and the budget constraint* $1 = q_1\psi_1^1 - q_2\,\phi_2^1$. *From the first order conditions the following equality must hold.*

$$\kappa_2(1) = \frac{q_2}{\alpha q_1}$$

Now, from the book-keeping map it must be true that

$$\begin{aligned}\kappa_2(1) &= \eta(2,1)\\ &= \frac{M^2(1)}{\alpha\psi_2^2}\\ &= \frac{1+\phi_1^2}{\alpha\psi_2^2}\end{aligned}$$

From agent 2's budget constraint

$$-1 = q_2\psi_2^2 - q_1\phi_1^2$$

we obtain

$$\begin{aligned}\kappa_2(1) &= \frac{1}{\alpha\psi_2^2} + \frac{q_2}{\alpha q_1} + \frac{1}{\alpha q_1\psi_2^2}\\ &= \frac{q_2}{\alpha q_1} + \frac{1}{\alpha\psi_2^2}\left(1+\frac{1}{q_1}\right)\end{aligned}$$

Again, to equalize both expressions we get that $q_1 = -1$, impossible in equilibrium. Hence this economy has no equilibrium.

Although the utility functions do not satisfy the monotonicity condition given by assumption U, it can be shown using identical indirect arguments as in [9] that if utility functions were

$$\begin{aligned}U^1 &= \ln\left[1+x^1(1)\right] + \delta_1 u^1\left(x_0^1\right)\\ U^2 &= \ln\left[1+x^2(2)\right] + \delta_2 u^2\left(x_0^2\right)\end{aligned}$$

with u^i strictly increasing, continuous and concave, and $\delta_i > 0$ and $\delta_i \to 0$, then for these preferences there is no equilibrium.

This example shows that absence of assumption C may imply no existence. However whether this property is generic (in the sense of being true for *almost all economies*) is not known. This is one of the key aspects to be studied in future research.

3 Extension of (standard) results in entrepreneurship economies.

This section modifies and proves traditional results in the GEI literature for entrepreneurship economies [3]. The main motivation for doing this is to see

[3]For a presentation of these results in standard production GEI economies see Magill and Quinzii [5] or Magill and Shafer [6] for a more advanced treatment.

whether one can easily extend results concerning risk evaluation and a version of the Modigliani-Miller theorem. The answer is that extensions are indeed not very difficult to make. However some definitions need to be changed to accommodate for the inclusion of unforeseen states.

First I introduce an extra assumption on preferences and technologies:

D Assume that all utility functions and production functions are C^k with $k \geq 2$, satisfying Inada conditions.

Then I can characterize the equilibrium through standard first order conditions. However, a difficulty arises because of the constraint $\phi^i \cdot \psi^i = 0$. This makes the budget constraint not convex. Still this is not a real problem. One can show (see the proof of existence of equilibrium, in [9]) that this is automatically satisfied if ignored. In other words, the problem without this constraint has a (unique) solution such that $\phi^i \cdot \psi^i = 0$, provided $\kappa_C = 1$. Since we consider equilibria with trade in all assets, then this condition holds.

3.1 Characterization of equilibrium asset prices.

The following proposition characterizes the non-arbitrage asset prices. From this point to the end of the paper I consider equilibria with trade in all assets.

Proposition 4 *In an equilibrium with trade in all assets, the corresponding formula for the price of asset j is given by the following expression.*

$$\begin{aligned} q_j &= \max_i \left\{ \sum_{s \in S^i} \bar{\pi}_s^i \left(\bar{x}^i\right) R_j\left(s\right) \kappa_j\left(s\right) \right\} \\ &= \min_i \left\{ \sum_{s \in S^i} \bar{\pi}_s^i \left(\bar{x}^i\right) R_j\left(s\right) \right\} \end{aligned}$$

where $\bar{\pi}_s^i\left(\bar{x}^i\right)$ is the equilibrium MRS for agent i between consumption at date 1, state s, and consumption at date 0.

Proof. See appendix. ∎

This generalizes the asset pricing formula for standard GEI economies. Note that the awareness of agents are crucial. It is also obvious to see that, if the agent i that solves $\max_i \left\{ \sum_{s \in S^i} \bar{\pi}_s^i \left(\bar{x}^i\right) R_j\left(s\right) \kappa_j\left(s\right) \right\}$ has a state space $S^i = C$ only then bankruptcy does not matter for pricing. As long as this consumer i is aware of some state such that some other consumers (who may be short in asset j) is not aware of, then the bankruptcy possibility enters in the expression for q_j.

3.2 Characterization of equilibrium allocations: the profit maximizing criterion

As it is usual in the literature, one can characterize the production decisions through a modified profit maximization decision. The proof of the following result is in the appendix.

Proposition 5 *The production decision in equilibrium can be characterized as the solution of :*

$$\max_{y_0^{k(i)}} \quad \Pi^i = -y_0^{k(i)} + \sum_{s \in S^i} \bar{\pi}_s^i \left(\bar{x}^i\right) f_s^i \left(y_0^{k(i)}\right)$$

Note that the profit maximization must be done only on the subset of states where the entrepreneur has awareness. This matters for the computation of the risk evaluation formula. This is done in the next sub-section.

3.3 Characterization of risk evaluation by entrepreneurs.

Another standard decomposition is the risk evaluation by entrepreneurs. To do this in our context, I add another assumption on preferences.

VNM Assume that the ex-ante, date 0 utility function of each consumer i takes the following form.

$$u^i = v^i\left(x_0^i\right) + \sum_{s \in S^i} \xi_s^i v^i\left(x^i(s)\right)$$

with

$$\sum_{s \in S^i} \xi_s^i = 1, \quad \xi_s^i \geq 0$$

with v^i being strictly increasing, strictly concave and C^k, $k \geq 2$.

Note that, unless $S^i = S^{i'}$ for all $i \neq i'$ it is impossible to make beliefs homogeneous. Therefore the risk evaluation by entrepreneurs must be in general subjective. Having said this, one can evaluate the expected *profits* or net present value of the production stream as follows.

$$\begin{aligned}
NPV^i &= -y_0^{k(i)} + E_i\left[\bar{\pi}^i f^i\right] \\
&= -y_0^{k(i)} + \sum_{s\in S_i} \xi_s^i \bar{\pi}_s^i\left(\bar{x}^i\right) f_s^i\left(y_0^{k(i)}\right) \\
&= -y_0^{k(i)} + \left(\sum_{s\in S_i} \xi_s^i \bar{\pi}_s^i\left(\bar{x}^i\right)\right)\left(\sum_{s\in S_i} \xi_s^i f_s^i\left(y_0^{k(i)}\right)\right) + \\
&\quad + \sum_{s\in S_i} \xi_s^i \left(\bar{\pi}_s^i\left(\bar{x}^i\right) - \left(\sum_{s\in S_i} \xi_s^i \bar{\pi}_s^i\left(\bar{x}^i\right)\right)\right)\left(f_s^i\left(y_0^{k(i)}\right) - \left(\sum_{s\in S_i} \xi_s^i f_s^i\left(y_0^{k(i)}\right)\right)\right)
\end{aligned}$$

which can be rewritten as

$$NPV^i = -y_0^{k(i)} + E_i\left(\bar{\pi}^i\right) E_i\left(f^i\left(y_0^{k(i)}\right)\right) + cov_i\left(\bar{\pi}^i; f^i\left(y_0^{k(i)}\right)\right)$$

This is close to the standard formula. However this cannot be reduced to the sum of the expected net present value (discounted at a riskless rate) and the covariance, at least in general. This is only possible if i is short on a riskless bond, or otherwise if $S^i = C$, or at least $S^i \subset S^l$, for all l such that l is short on a riskless bond. Only in this case we can state that

$$E_i\left(\bar{\pi}^i\right) = q_r \equiv \frac{1}{1+r_r}$$

where q_r is the price of a bond paying one unit of consumption regardless of the state and r_r is the correspondent riskless rate. Only then we have

$$NPV^i = -y_0^{k(i)} + \frac{E_i\left(f^i\left(y_0^{k(i)}\right)\right)}{1+r_r} + cov_i\left(\bar{\pi}^i; f^i\left(y_0^{k(i)}\right)\right)$$

The reason of the (strong) condition is the presence of bankruptcy for those states not foreseen by agents who are short in assets. If i is a purchaser of this bond and is aware of some states that some other agent i' is not aware of, and this agent is short on this bond, then the last expression may break down due to $\kappa < 1$.

Another straightforward though essential feature is the fact that entrepreneur i may be aware of *too little* states. In particular the agent may be aware of only one state in the future. In this case this implies no subjective risk and the only variable to look at is the discounted expected production flow. As one can see, this can be a way of analyzing *pessimistic* or *optimistic* entrepreneurs. Those who are only aware of *bad* states could be seen as pessimistic entrepreneurs. They would invest too little in the production technology. Those who are only aware of extremely *good* states could be called optimistic, which easily would lead to *overinvestment*. However, my goal here is not to pursue an efficiency analysis due to the unawareness of consumers. I leave this as a task for future research.

3.4 Indeterminacy of financial policies in entrepreneur economies.

The well known Modigliani - Miller theorem [10] has been the center of the discussion for at least thirty years. A broad literature appeared during the seventies and eighties. This sub-section attempts to re-check the validity of this result under the assumption of unawareness in entrepreneurship economies.

The main conclusion is that the MM result is still valid under certain modifications. The unawareness assumption demands some minor changes to the proof and statement. However the idea remains as long as entrepreneurship economies are concerned.

Assume that the entrepreneurship $k(i)$ takes decision on production through the following criterion.

$$\max \sum_{s \in S^i} \pi_s^i D_s^{k(i)} + D_0^{k(i)}$$

where

$$\begin{aligned} D_0^{k(i)} &= -y_0^{k(i)} - q \cdot \left(\alpha^i - \beta^i\right) \\ D_s^{k(i)} &= f_s^{k(i)}\left(y_0^{k(i)}\right) + \left(\kappa(s) \otimes R(s)\right) \cdot \alpha^i - R(s) \cdot \beta^i \\ \alpha^i, \beta^i &\geq 0 \end{aligned}$$

I call this the *production* problem. Here $D_0^{k(i)}$ represents the *dividends* that $k(i)$ obtains in period 0 and $D_s^{k(i)}$ those obtained in period 1, state s. In fact $\bar{D}_0^{k(i)}$ really represents what the entrepreneur invests in her own production technology. It also includes the financing policy as entrepreneur, given by the expression $q \cdot \left(\alpha^i - \beta^i\right)$. Here α^i represents the asset purchases and β^i the sales that the entrepreneur decides to finance production. It can be shown again that at the solution $\alpha^i \cdot \beta^i = 0$. Note also that I allow $D_s^{k(i)}$ and $D_0^{k(i)}$ to be either positive or negative.

Let $\left(\bar{y}^{k(i)}, \bar{\alpha}^i, \bar{\beta}^i\right)$ be the solution to this problem. Let $\bar{D}$ denote the dividends arising from the solution. The entrepreneur as a consumer solves in period 0:

$$\max \quad u^i\left(x_0^i, \left(x^i(s)\right)_{s\in S^i}\right)$$

subject to

$$\begin{aligned}
x_0^i - \omega_0^i &\leq \bar{D}_0^{k(i)} - q\left(\phi^i - \psi^i\right) \\
x^i(s) - \omega^i(s) &\leq \bar{D}_s^{k(i)} + \left(\kappa(s) \otimes R(s)\right)\cdot\phi^i - R(s)\cdot\psi^i, \quad s \in S^i \\
\phi^i, \psi^i &\geq 0
\end{aligned}$$

and for states $s \notin S^i$ the agent just consumes

$$x^i(s) = \max\left\{0; \omega^i(s) + \bar{D}_s^{k(i)} + \left(\kappa(s)\otimes R(s)\right)\cdot\phi^i - R(s)\cdot\psi^i\right\}$$

This is called the *consumption* problem.

I define an entrepreneurship equilibrium.

Definition 6 *An entrepreneurship equilibrium is an allocation* $\left(\bar{y}^{k(i)}, \bar{\alpha}^i, \bar{\beta}^i, \bar{x}^i, \bar{\phi}^i, \bar{\psi}^i\right)$ *and asset prices* $\bar{q}$ *such that*

- *The vector* $\left(\bar{y}^{k(i)}, \bar{\alpha}^i, \bar{\beta}^i\right)$ *solves the entrepreneur's production problem, taking* $\bar{q}$ *and* $\bar{x}^i$ *as given.*
- *The vector* $\left(\bar{x}^i, \bar{\phi}^i, \bar{\psi}^i\right)$ *solves the agent's consumption problem, taking* $\bar{q}$ *and* $\bar{D}^{k(i)}$ *as given.*
- *Market clearing holds*

$$\begin{aligned}
\sum_{i=1}^{I}\left(\bar{\phi}^i + \bar{\alpha}^i\right) &= \sum_{i=1}^{I}\left(\bar{\psi}^i + \bar{\beta}^i\right) \\
\sum_{i=1}^{I}\left(\bar{x}^i - \omega^i\right) &= \sum_{i=1}^{I}\bar{y}^{k(i)}
\end{aligned}$$

This is the standard definition of an equilibrium with separation of decision policies. With this it is possible to show the following result.

Proposition 7 *In an entrepreneurship economy* $\left(\bar{x}^i, \bar{y}^i, \bar{\phi}^i, \bar{\psi}^i, \bar{\alpha}^i, \bar{\beta}^i\right)$ *is an entrepreneurship equilibrium allocation if and only if* $\left(\bar{x}^i, \bar{y}^i, \widehat{\phi}^i, \widehat{\psi}^i\right)$ *is a competitive equilibrium allocation with* $\widehat{\phi}^i = \bar{\phi}^i + \bar{\alpha}^i$ *and* $\widehat{\psi}^i = \bar{\psi}^i + \bar{\beta}^i$.

Proof. See Appendix. ■

This last proposition shows that financial policies do not matter in entrepreneurship economies provided that dividends do not include limited liability properties. Hence unawareness does not change radically the sense of the Modigliani - Miller theorem, provided that bankruptcy is *not* strategic or foreseen.

4 An example with *creation* of a new good.

So far we have seen an economy such that entrepreneurs owned technologies that produced the same commodity that agents faced in period 0. In other words, technologies only were able to produce the same type of good. In this case agents and specially entrepreneurs were only unaware of the productivity of their technology in some states. On the other hand, as Arrow and Hahn [1] emphasized, it is impossible in practice for *innovations* and *inventions* to account for all possible future contingencies. It is then hard to describe all possible outcomes that an invention could carry on. What this section attempts to do is to present a very simple example with a technology that creates a new good in period 1, at a certain state. The point of this section is to illustrate whether unawareness of investors (non-owners) affect the price of the new good in the case of successful innovation, as well as asset prices.

Suppose an economy with two agents (or a continuum of two types). There are two states of the world in period 1. Consumer 1 owns a production technology. This technology produces a second commodity only in state 2, but not in state 1. Hence at state 2 in period 1 there are two physical commodities. Let the endowments of agent 1 be

$$\omega_0^1 = 0, \quad \omega_1^1(1) = 1, \quad \omega_1^1(2) = 1, \quad \omega_2^1(2) = 0$$

The production technology is given by the production function

$$y_2^1(s) = \begin{cases} \sqrt{y_0^1}, & \text{if } s = 2 \\ 0, & \text{otherwise} \end{cases}$$

where y_0^1 is measured in terms of good 1 and $y_2^1(2)$ denotes the units of good 2 obtained through this technology.

On the other hand, consumer 2 does not own any technology. Her endowments are

$$\omega_0^2 = 1, \quad \omega_1^2(1) = 0, \quad \omega_1^2(2) = 1, \quad \omega_2^2(2) = 0$$

I will consider a special case. The entrepreneur (agent 1) is always *optimistic* with respect to her technology. This is captured through the assumption $S^1 = \{2\}$. Given this I assume the following utility functions for agent 1.

$$u^1 = \ln x_1^1(2) + \ln x_2^1(2)$$

(note again that she does not care of consumption at date 0). If state 1 is realized the utility function is just $u_1^1(x_1^1(1))$. I am not assuming any particular functional form since she just consumes what is available. I just need that $|u_1^1(0)| < \infty$. For agent 2 the utility functions will be

$$u_2^2 = u^1$$

in state 2 and

$$u_1^2 = \ln x_1^2(1)$$

in state 1. In period 0 is

$$u_0^2 = \ln x_0^2$$

There is one riskless asset. Payoffs are written in good 1 (the numeraire in state 2). The asset pays one unit of good 1 per unit of contract.

I consider two subcases: one in which $S^2 = \{1\}$ and the other when $S^2 = \{2\}$.

4.1 The case when 2 is unaware of the new good

This is the case when the only relevant state for agent 2 is the one without the new commodity. On the other hand, agent 1 is only aware of the success of her invention. In this case, it is natural to search for an equilibrium (if it exists) such that $\psi^1 > 0$. This is stating that agent 1, the entrepreneur, uses the riskless bond to borrow funds from agent 2 to run the invention. The problem for agent 1 in this case reduces to

$$\max \quad \ln x_1^1(2) + \ln x_2^1(2)$$

subject to

$$x_1^1(2) + p_2 x_2^1(2) \leq 1 + p_2\sqrt{y_0^1} + \phi^1 - \psi^1$$

(where I omit the parenthesis for p_2 for obvious reasons) and

$$(\psi^1 - \phi^1) \cdot q = y_0^1$$

The FOC with respect to ψ^1 , assuming that this is positive, is:

$$-1+\frac{p_2}{2}\left(\sqrt{\frac{q}{\psi^1}}\right)=0$$

which gives

$$\psi^{1*}=\frac{qp_2^2}{4}$$

and then

$$y_0^1=\frac{q^2p_2^2}{4} \quad \Rightarrow \quad y_2^1(2)=\frac{qp_2}{2}$$

From the rest of the first order conditions it is straightforward to get

$$\begin{aligned} x_1^1(2) &= \frac{1+\frac{qp_2^2}{4}}{2} \\ x_2^2(2) &= \frac{1+\frac{qp_2^2}{4}}{2p_2} \end{aligned}$$

In this equilibrium agent 2 must purchase the riskless bond. This must solve the following maximization problem.

$$\max \quad \ln x_0^2+\ln x_1^2(1)$$

subject to

$$\begin{aligned} x_0^2 &= 1-q\left(\phi^2-\psi^2\right) \\ x_1^2(1) &= \kappa(1)\phi^2-\psi^2 \end{aligned}$$

The solution to the problem is

$$\phi^{2*}=\frac{1}{2q}$$

Note that this is independent of $\kappa(1)$. Then, by imposing market clearing in the bond market, we get

$$q^*=\frac{\sqrt{2}}{p_2^*}$$

Then we have

$$\psi^{1*}=\phi^{2*}=\frac{p_2^*}{2\sqrt{2}}$$

and also

$$y_0^{1*} = \frac{1}{2}, \quad y_2^1(2) = \frac{\sqrt{2}}{2}$$

In state 2, agent 2 solves

$$\max \quad \ln x_1^2(2) + \ln x_2^2(2)$$

subject to

$$x_1^2(2) + p_2 x_2^2(2) = 1 + \phi^{2*}$$

The optimal demand functions are

$$x_1^2(2) = \frac{1 + \frac{p_2^*}{2\sqrt{2}}}{2}; \quad x_2^2(2) = \frac{1 + \frac{p_2^*}{2\sqrt{2}}}{2p_2}$$

Imposing market clearing in the market for good 2 we obtain that the equilibrium price of good 2 is

$$\begin{aligned} p_2^* &= \frac{2}{\left(\frac{3}{4}\right)\sqrt{2} - \frac{1}{2\sqrt{2}}} \\ &= \sqrt{8} \end{aligned}$$

which is clearly positive. Next, I get $\kappa(1)$. To do this, note that

$$M^1(1) = \min\{\psi^{1*}, 1\}$$

But in this equilibrium

$$\psi^{1*} = 1$$

Hence $M^1(1) = 1$. By definition $\kappa(1) = \eta(1,1) = 1$ (there is no bankruptcy in equilibrium, although consumer 1 gets 0 in state 1). The equilibrium is as follows

p_2	q	$\phi^2 = \psi^1$
$\sqrt{8}$	$\frac{1}{2}$	1

$x_1^1(1)$	$x_1^1(2)$	$x_2^1(2)$	x_0^2	$x_1^2(1)$	$x_1^2(2)$	$x_2^2(2)$
0	1	$\frac{\sqrt{2}}{4}$	$\frac{1}{2}$	1	1	$\frac{\sqrt{2}}{4}$

and equilibrium output is $y_2 = 0.5\sqrt{2}$. In this equilibrium there is no default (here $\kappa(s) = 1$ for $s = 1, 2$) although agent 1 has just the amount of good 1 to repay her debt in state 1. In state 2 both agents have exactly the same ex-post utility, although agent 2 was not aware of this state. The next subsection modifies this assumption to see whether any change results.

4.2 Case where 2 is aware of the new good

This version of the mentioned economy assumes

$$S^i = \{2\}$$

for both $i = 1, 2$. This is the case when both agents are aware (only) of the success of the invention. The idea is to compute the new equilibrium to compare the outcomes.

In this new economy the problem for agent 1 is identical to the one we got before. Hence we get the same expressions for the demands:

$$\begin{aligned} x_1^1(2) &= \frac{1+\frac{qp_2^2}{4}}{2} \\ x_2^1(2) &= \frac{1+\frac{qp_2^2}{4}}{2p_2} \end{aligned}$$

The problem for agent 2 is different. In period 0 she solves the following problem:

$$\max \quad \ln x_0^2 + \ln x_1^2(2) + \ln x_2^2(2)$$

subject to

$$\begin{aligned} x_0^2 &= 1 - q\phi^2 \\ x_1^2(2) + p_2 x_2^2(2) &= 1 + \phi^2 \end{aligned}$$

In this equilibrium $\psi^1 > 0$ and hence $\kappa(2) = 1$. The solution to this problem gives the following optimal bond holding:

$$\phi^{2**} = \frac{2-q}{3q}$$

Imposing market clearing in the asset market implies the following expression for the equilibrium asset price:

$$q^{**} = \left(\frac{-1+\sqrt{1+6p_2^2}}{p_2^2}\right)\left(\frac{2}{3}\right)$$

The optimal consumption for agent 2 in state 2 of good 1 is

$$\begin{aligned} x_1^{2**}(2) &= \frac{1+\phi^{2**}}{2} \\ &= \frac{1}{2} + \frac{2-\left(\frac{-1+\sqrt{1+6p_2^2}}{p_2^2}\right)\left(\frac{2}{3}\right)}{6\left(\frac{-1+\sqrt{1+6p_2^2}}{p_2^2}\right)\left(\frac{2}{3}\right)} \end{aligned}$$

Solving this:

$$x_1^{2**}(2) = \frac{1}{2} + \frac{1}{6}\left[\frac{3p_2^2 + 1 - \sqrt{1 + 6p_2^4}}{\sqrt{1 + 6p_2^4} - 1}\right]$$

Imposing market clearing in this market:

$$\frac{1}{2} + \frac{1}{6}\left[\frac{3p_2^2 + 1 - \sqrt{1 + 6p_2^2}}{\sqrt{1 + 6p_2^2} - 1}\right] + \frac{1 + \frac{qp_2^2}{4}}{2} = 2$$

Solving this we get $p_2 = \sqrt{8}$, the same equilibrium price! In fact this gives exactly the same equilibrium outcome as in the case where $S^2 = \{1\}$.

This example shows that the intuition saying that more awareness would imply a higher $y_2(2)$ is not satisfied. In the last case the agent that purchases the bond is aware of the new good. However she acts in the same way as when she is not aware of it. The main lesson is that there may be some cases in which awareness may not matter in terms of how much is invested for innovations. How much this outcome depends on the value of endowments, state spaces and utility functions is something important to know, but left for future research.

It is important to see that agent 2 in the first case is not aware of state 1 and even not aware of the demand function of agent 1. Recall that, given the competitive markets assumption, they only observe prices and then decide how much of the asset to buy or sell short. But they do not know other people's preferences and trades. This is relevant since otherwise agent 2 could infer something about state 2 if she observes agent 1's trade. But then agent 2 should be aware of state 2. To avoid this problem we never allow any agent to be aware of other agent's characteristics or trades.

5 Towards Partnership Economies

Although entrepreneurship economies are somehow the simplest case to start analyzing unawareness issues in the GEI literature, it is not necessarily the most realistic one. Production is usually organized in firms with multiple owners or partners. There is a vast literature on GEI in partnership economies in the standard framework. My goal in this section is to present the preliminaries of an extension of this analysis to the unforeseen contingencies case. This includes a modified definition of a partnership equilibrium and some results regarding the objective of the firm.

Modify the economy of section 2 in the following way. Assume that each of the K firms could be exploited by all I agents. Instead of having some of the agents being individual owners, each agent could purchase a fraction θ_i^k of firm k, with $\theta_i^k \geq 0$ and $\sum_{i=1}^{I} \theta_i^k = 1$. Hence the budget constraint faced by agent i in period 0 is the following one.

$$x_0^i = \omega_0^i + \sum_{j=1}^{J} \left(\psi_j^i - \phi_j^i\right) q_j - \sum_{k=1}^{K} \theta_i^k y_0^k \tag{2}$$

$$x^i(s) = \omega^i(s) + \sum_{j=1}^{J} R_j(s)\,\kappa_j(s)\,\phi_j^i - \sum_{j=1}^{J} R_j(s)\,\psi_j^i + \sum_{k=1}^{K} \theta_i^k f_s^k\left(y_0^k\right), \quad s \in S^i \tag{3}$$

and for $s \notin S^i$ we have just:

$$x^i(s) = \max\left\{\omega^i(s) + \sum_{j=1}^{J} R_j(s)\,\kappa_j(s)\,\phi_j^i - \sum_{j=1}^{J} R_j(s)\,\psi_j^i + \sum_{k=1}^{K} \theta_i^k f_s^k\left(y_0^k\right); 0\right\} \tag{4}$$

The definition of $\kappa_j(s)$ follows exactly the same expressions as in the entrepreneurship case.

Hence I can define a (Pareto) competitive equilibrium for this partnership economy in the usual way.

Definition 8 *An (ex-ante) Pareto Partnership Equilibrium is a set of asset prices* $\bar{q}$ *and allocations* $\left(\bar{x}^i, \bar{\theta}_i, \bar{\phi}^i, \bar{\psi}^i\right)_{i=1}^{I}$, $\left(\bar{y}^k\right)_{k=1}^{K}$, *such that*

- *For each agent i, the allocation* $\left(\bar{x}_0^i, \left(\bar{x}^i(s)\right)_{s\in S^i}, \bar{\theta}_i, \bar{\phi}^i, \bar{\psi}^i\right)_{i=1}^{I}$ *solves*

  $$\max \quad u^i\left(x_0^i, \left(x^i(s)\right)_{s\in S^i}\right)$$

  subject to 2 and 3, and the non-negativity constraints (as in the entrepreneurship economy).

- *For each agent i, the allocation* $\left(\bar{x}^i(s)\right)_{s\notin S^i}$ *satisfies 4 with* $\left(\bar{\theta}_i, \bar{\phi}^i, \bar{\psi}^i\right)$ *given*

- *(Non-ex-ante Pareto improvement) For each k, there is no* y_0^k *such that* $u^i\left(\bar{x}_0^i - \bar{\theta}_i\left(y_0^k - \bar{y}_0^k\right); \left(\bar{x}^i(s) + \bar{\theta}_i\left(f_s^k\left(y_0^k\right) - f_s^k\left(\bar{y}_0^k\right)\right)\right)_{s\in S^i}\right) \geq u^i\left(\bar{x}_0^i, \left(\bar{x}^i(s)\right)_{s\in S^i}\right)$.

- *Market clearing holds*

$$\sum_{i=1}^{I}\left(\bar{\phi}^{i}-\bar{\psi}^{i}\right) = 0$$

$$\sum_{i=1}^{I}\bar{x}_{0}^{i}+\sum_{k=1}^{K}\bar{y}_{0}^{k} = \sum_{i=1}^{I}\omega_{0}^{i}$$

$$\sum_{i=1}^{I}\bar{x}^{i}(s) = \sum_{k=1}^{K}\bar{y}_{1}^{k}(s)+\sum_{i=1}^{I}\omega^{i}(s)$$

where $\bar{y}_{1}^{k}(s)=f_{s}^{k}\left(\bar{y}_{0}^{k}\right)$.

The only important difference with respect to the standard definition is the fact that the *no Pareto improvement* condition only requires that there cannot be an $ex-ante$ improvement in production. There is no ex-post non improvement condition. It seems natural to assume so since production decisions are taken at date 0. This is key to characterize production decisions through net present value maximization. The result is presented in the following proposition.

Proposition 9 *Assume condition U. The Non-ex-ante Pareto improvement is equivalent to the following:*

i There is no $y_{0}^{k}>0$ *such that*

$$-y_{0}^{k}+\sum_{s\in S^{i}}\bar{\pi}_{s}^{i}f_{s}\left(y_{0}^{k}\right)>-\bar{y}_{0}^{k}+\sum_{s\in S^{i}}\bar{\pi}_{s}^{i}f_{s}\left(\bar{y}_{0}^{k}\right)$$

for every i such that $\theta_{k}^{i}>0$.

ii There are non-negative scalars $\left(\alpha^{1},...,\alpha^{I}\right)$ *such that*

$$\bar{y}_{0}^{k}\in\arg\max\left\{\sum_{i\in I_{k}}\alpha_{k}^{i}\bar{\theta}_{k}^{i}\left[\sum_{s\in S^{i}}\bar{\pi}_{s}^{i}f_{s}^{k}\left(y_{0}^{k}\right)-y_{0}^{k}\right]\right\}$$

Proof. See Appendix ■

This result is a modification of the standard NPV characterization. The proof needs some modification due to dimensionality issues. Concretely, the following new version of the Minkowski-Farkas lemma needs to be shown in order to prove the equivalence between i and ii.

Proof.

Lemma 10 *Let $S^* \equiv \cup_i S^i$, let $\bar{\pi} \in \Re^{S^*}$, and let $\bar{\pi}^i \in \Re^{S^i}$. The inequality $\sum_{s\in S^*} \bar{\pi}_s y_s - \bar{\pi}_0 y_0 \geq 0$ is a consequence of the system of inequalities $\sum_{s\in S^i} \bar{\pi}^i_s y_s - y_0 \geq 0$ if and only if there are non-negative real numbers $\alpha^1, ..., \alpha^I$ such that*

$$\bar{\pi} = \sum_i \alpha^i \widehat{\pi}^i$$

where

$$\begin{aligned} \widehat{\pi}^i &\equiv \left(-1, \bar{\pi}^i, 0\right) \\ 0 &\in \Re^{S^*-S^i} \end{aligned}$$

■

The proof of this is in the appendix. In spite of the technical changes, the production characterization result is still valid in the unawareness case. That is, in equilibrium the production allocation must maximize the subjective net present value for each owner.

On the other hand, the result of unanimity in production decisions under constant returns to scale, together with multiplicative uncertainty in the production function still holds when unawareness is considered. The following proposition shows this.

Proposition 11 *Suppose that $f^k_s\left(y^k_0\right) = \rho\left(s\right) y^k_0$ for all k. Assume moreover assumptions C, U and consider an equilibrium with trade in all assets. Then all partners of each firm k agrees on the optimal production decision.*

This is an extension of the well-known Diamond's [2] result. The multiplicative uncertainty assumption still plays the same role with unawareness as in the standard partnership economies. This is relevant since the usual problem arising with this type of equilibrium concept is the intrinsic non-agreement in production decisions. This proposition then states that unawareness does not change the main property of multiplicative shocks.

6 Concluding Remarks and Future Research

The main message of the paper is that unawareness does not seem to be key to change classical results in GEI with production. Still along this paper assumption C was assumed to hold all the time. Dropping this assumption is essential for future research, still in pure exchange economies. The counterexample presented at the end of section 2 may not be generic. As it was the rule in this literature, generic existence of equilibria in GEI economies were shown after the famous counterexample by Hart [4]. In a similar fashion, it is essential to study whether similar techniques could be applied to GEI economies with unawareness, in order to understand the role of assumption C. This is important because, if assumption C were shown to be generically essential to get existence, this would mean that studying unawareness in the competitive framework may not be the most interesting market structure to analyze. If, in order to get consistency of the model, all agents almost always must be aware of at least the subspace spanned by all assets, then we would finally be imposing in some sense too much awareness. In reality many investors may not be even aware of what the assets could span in the future. Much discussion must be done in order to improve our understanding of the relationship between these two concepts.

Another lesson from this paper is the fact that two economies with different awareness conditions may lead to the same equilibrium outcome. A more profound analysis is needed to understand the generality of this result. This is important because models like this could provide some clarifications of differences in investment and production decisions across countries. It seems sensible to think that some form of awareness may affect those differences. However the example presented here clearly shows that unforeseen contingencies may not imply any heterogeneity. Whether this example can be generalized is left for future research.

The ownership structures considered in this paper are rather quite simple. Therefore considering corporations is another important step to take. However, some new complications may arise. In a stock market equilibrium, the asset structure includes shares of the firms. Assets in this case are the shares of the K firms together with J bonds. The question now is whether assumption C should be modified to include the first $J+K$ states of the economy rather than J. If the former is the case, it would make the conditions for existence even stronger. This may show again some weakness of the concept of competitive equilibrium with unawareness, since making the asset structure richer would demand agents being aware of more states. Also, the issue of limited liability in corporations could introduce a major complication in the analysis. This is another key point to study in the future.

Two more issues need some late discussion. All these economies last for two periods. Note, in section 5, that agent 2 has well defined preferences on both goods, when state 2 is realized, even when she is not ex-ante aware of it. This is clearly strong. The interpretation of this assumption could be the following. The outcome of some *learning process* by agent 2 after seeing an unforeseen state is the period 1, state 2 utility function. However that process is clearly not modelled. In a sense periods 0 and 1 could be thought of two steady states. It is still very important to think about modelling learning processes with the unawareness structure [4]. This is indeed a major challenge for extending the work in [7] and [8] to dynamic contexts. Perhaps recent papers in foundations of adaptive learning such as Easley and Rustichini [3] could be a good starting point.

The second point is the efficiency discussion. The reader can see that I avoided that problem. This is not a trivial task. The main question is to know what is the relevant Pareto efficiency concept. It is clear that it can never be Pareto efficient in the usual sense, due to the unforeseen states. The discussion would focus on what kind of constraints are needed to be added to the Pareto efficiency criterion. In other words, how one characterizes the conditioned Pareto problem is the question. Because of potentially heterogeneous awareness among agents, defining a sensible social planner problem is not easy. Probably this could be considered a major challenge for entire field of economics of uncertainty.

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7 Appendix: Proofs.

Proof of Proposition 2. The proof is essentially the replication of the one in [9] The only difference is the production technology to the budget constraint. I then show how the original proof should be modified. The rest is as in the cited paper.

The date 0 budget constraint is

$$B^i = \left\{ \begin{array}{c} x^i \in \Re^{S^i+1} : x_0^i \leq \omega_0^i + \left(\psi^i - \phi^i\right) \cdot q - y_0^{k(i)} \\ x_1^i(s) \leq \omega_1^i(s) + (R \otimes \kappa)_s \cdot \phi^i - R_s \cdot \psi^i + y_1^{k(i)}(s) \\ \phi^i \cdot \psi^i = 0; \quad y_1^{k(i)}(s) = f_s^{k(i)}\left(y_0^{k(i)}\right) \end{array} \right\}$$

Then one can modify lemma 3.3.3 in [9] by the following.

Claim 12 *The correspondence* $(q, \kappa) \rightarrow B^i(q, \kappa)$ *is compact valued on* $Q_{S^i}(\kappa)$, $\kappa \in [0,1]^J$ *(where* $Q_{S^i}(\kappa)$ *is defined in [9], pp. 267) and continuous.*

The proof is at the end of this demonstration. Lemma 3.3.5 in [9] then follows. Note that the optimal $y_0^{k(i)}$ is also unique because $f_s^{k(i)}$ is strictly concave. The proof of this lemma also shows that the constraint $\phi\psi = 0$ is in fact non-binding at the equilibrium. This ensures that the optimal policy correspondence is *uhc* and convex-valued (in this case I also use the continuity assumption on f_s^k). On the other hand, because there is only one commodity, there is no real maximization problem in states $s \notin S^i$. Hence it is immediate that $x^i(s)$ for $s \notin S^i$ is continuous.

Let n be a natural number. If we limit q to lie on the set $\bar{Q}^n$ defined in the original proof. Let $q \leq M$, with M positive and large enough so that the intersection $[0, M]^J \cap \bar{Q}^n \neq \emptyset$ for every n. Then q is on a compact set. Therefore, a very similar argument to the one in lemma 3.3.9 can be given to show the existence of a fixed for the excess demand correspondence, the *market maker* correspondence (as defined in pp. 274) and the book keeping correspondence (as defined in the text).

Finally, when $n \to \infty$, we know that the sequence of fixed points $(q_n, \kappa_n)_{n=1}^{\infty}$ must have a convergent subsequence (because of boundedness of domain). Focus on those n_k such that $(q_{n_k}, \kappa_{n_k})_{k=1}^{\infty}$ converges. By the same argument as in Lemma 3.5.1, the sequence of allocations $\left(x_{n_k}, y_{n_k}, \phi_{n_k}, \psi_{n_k}\right)_{k=1}^{\infty}$ is bounded and then it has a convergent subsequence. Invoking similar reasonings as in lemmas 3.6.1 and 3.6.2 it can be shown that the limit of $(q_{n_k}, \kappa_{n_k})_{k=1}^{\infty}$ is part of an equilibrium. Finally, following the same lines of the proof of lemma 3.6.6 (or equivalently, the proof of Theorem in Werner [14]) implies that the limit of $\left(x_{n_k}, y_{n_k}, \phi_{n_k}, \psi_{n_k}\right)_{k=1}^{\infty}$ is indeed an equilibrium allocation. ∎

Proof of claim 12. That $B^i(q, \kappa)$ is closed follows from the Y assumption and the standard sequence arguments. Boundedness follows from the same arguments as in the original proof of lemma 3.3.6 (ii), with the additional fact that y_0^k is bounded. By the non-arbitrage asset price set, $Q_{S^i}(\kappa)$, defined in [9], it is clear that

$$\lambda \cdot (R \otimes \kappa) \cdot \phi - \lambda \cdot R \cdot \psi \leq q \cdot (\phi - \psi)$$

Then, if we multiply each inequality in B^i for $s \in S^i$ by λ_s and the we sum over states $s \in S^i$ and over dates, we get

$$x_0^i + \sum_{s \in S^i} \lambda_s x^i(s) + y_0^{k(i)} - \sum_{s \in S^i} \lambda_s f_s^{k(i)}\left(y_0^{k(i)}\right) \leq \omega_0^i + \sum_{s \in S^i} \lambda_s \omega^i(s)$$

By assumption Y, the consumption allocations and production allocations in B^i are bounded. This shows compactness of B^i. The proof of l.h.c. and u.h.c. is identical to the proof of lemma 3.3.3 (iii) and Corollary 3.3.4 in page 276. In our case we need to use continuity of $f_s^k\left(y_0^k\right)$ to ensure that the sequence

$\left\{x_n^i, y_n^{k(i)}, \phi_n^i, \psi_i^n\right\}_{n\geq 1}$ is bounded when $q_n \to q$ and $\kappa_n \to \kappa$. Hence claim 11 is shown. ■

Proof of Proposition 4. The first order conditions of the CP^i problem are

$$\begin{aligned}
\frac{\partial u^i}{\partial x_0^i} &= \chi_0^i \\
\frac{\partial u^i}{\partial x^i(s)} &= \chi_s^i, \quad s \in S^i \\
-\chi_0^i q_j + \sum_{s\in S^i} \chi_s^i R_j(s)\kappa_j(s) &\leq 0 \quad j = 1, ..., J \\
\chi_0^i q_j - \sum_{s\in S^i} \chi_s^i R_j(s) &\leq 0 \quad j = 1, ..., J \\
-\chi_0^i + \sum_{s\in S^i} f_s'\left(y_0^{k(i)}\right) &= 0
\end{aligned}$$

where the χ^i 's denote the Lagrange multipliers of the budget constraints. Here at least one of the two inequalities above holds with strict equality. From the third and the fourth inequalities is clear that, for all i:

$$\sum_{s\in S^i} \pi_s^i\left(\bar{x}^i\right) R_j(s)\kappa_j(s) \leq q_j \leq \sum_{s\in S^i} \pi_s^i\left(\bar{x}^i\right) R_j(s)$$

where

$$\pi_s^i\left(\bar{x}^i\right) \equiv \frac{\left(\frac{\partial u^i}{\partial x^i(s)}\right)\left(\bar{x}^i\right)}{\left(\frac{\partial u^i}{\partial x_0^i}\right)\left(\bar{x}^i\right)}$$

and $\bar{x}^i$ is the equilibrium consumption allocation for agent 1. Since the equilibrium considered here involves trading in all assets, it is the case that there is some i such that

$$\sum_{s\in S^i} \pi_s^i\left(\bar{x}^i\right) R_j(s)\kappa_j(s) = q_j$$

and also some other i' such that

$$q_j = \sum_{s\in S^{i'}} \pi_s^{i'}\left(\bar{x}^{i'}\right) R_j(s)$$

These two agents are those included in the statement of the proposition. That is, $i \in \arg\max \sum_{s\in S^i} \pi_s^i\left(\bar{x}^i\right) R_j(s)\kappa_j(s)$ and $i' \in \arg\min \sum_{s\in S^i} \pi_s^i\left(\bar{x}^i\right) R_j(s)$. This concludes the proof. ■

Proof of Proposition 5. The FOC of the profit maximization problem (which is sufficient and necessary due to strict concavity of f_s^i) is

$$-1 + \sum_{s \in S^i} \bar{\pi}^i \left(\bar{x}^i\right) f_s^{i\prime} \left(y_0^{k(i)}\right) = 0$$

By definition of $\bar{\pi}^i \left(\bar{x}^i\right)$ and assumption U this is equivalent to

$$-\frac{\partial u^i}{\partial x_0^i} + \sum_{s \in S^i} \left(\frac{\partial u^i}{\partial x^i (s)}\right) f_s^{i\prime} \left(y_0^{k(i)}\right) = 0$$

which is identical to the last equation in the characterization of the optimal choice in equilibrium by agent i given in the last proof. ■

Proof of Proposition 7. The FOC of the production problem (in addition to constraints) are

$$\begin{aligned} \sum_{s \in S^i} f_s^{k(i)\prime} \left(y_0^{k(i)}\right) &= 1 \\ -q_j + \sum_{s \in S^i} \pi_s^i \left(\bar{x}_i\right) R_j (s) \kappa_j (s) &\leq 0 \\ q_j - \sum_{s \in S^i} \pi_s^i \left(\bar{x}_i\right) R_j (s) &\leq 0 \end{aligned}$$

On the other hand, the first order conditions from the consumption problem (besides the constraints) are also

$$\begin{aligned} -\left(\frac{\partial u^i}{\partial x_0^i}\right) q_j + \sum_{s \in S^i} \left(\frac{\partial u^i}{\partial x^i (s)}\right) R_j (s) \kappa_j (s) &\leq 0 \\ \left(\frac{\partial u^i}{\partial x_0^i}\right) q_j - \sum_{s \in S^i} \left(\frac{\partial u^i}{\partial x^i (s)}\right) R_j (s) &\leq 0 \end{aligned}$$

It is obvious from the definition of $\pi_s^i \left(\bar{x}_i\right)$ that these last two inequalities are identical to the last two inequalities from the production problem. This implies that there are two expressions which are redundant. Hence, when computing an entrepreneurship equilibrium, one can assume without loss of generality that α^i, β^i, ϕ^i and ψ^i are such that $\left(\alpha^i + \phi^i\right) \cdot \left(\beta^i + \psi^i\right) = 0$. For example, take just $\alpha^i = \beta^i = 0$. This satisfies all the conditions. We see immediately that these allocations satisfies all the optimality conditions in the standard competitive equilibrium.

For the reverse, just fix α^i, β^i such that $\alpha^i \cdot \beta^i = 0$ and such that , if we define $\phi^i \equiv \widehat{\phi}^i - \alpha^i$ and $\psi^i \equiv \widehat{\psi}^i - \alpha^i$, then $\phi^i \cdot \psi^i = 0$. ■

Proof of Proposition 9. To simplify let NPIC denote the Non-Pareto improvement condition. Also, define $I_k \equiv \{i \in I \quad | \quad \theta_k^i > 0\}$. The first two parts follow standard lines. The last part is slightly different.

Suppose that NPIC holds but i does not hold. This implies that there exists y_0^k such that

$$-y_0^k + \sum_{s \in S^i} \bar{\pi}_s^i f_s\left(y_0^k\right) > -\bar{y}_0^k + \sum_{s \in S^i} \bar{\pi}_s^i f_s\left(\bar{y}_0^k\right)$$

for every i in I_k. Hence this implies

$$-y_0^k \bar{\theta}_k^i + \sum_{s \in S^i} \bar{\pi}_s^i f_s\left(y_0^k\right) \bar{\theta}_k^i > -\bar{y}_0^k \bar{\theta}_k^i + \sum_{s \in S^i} \bar{\pi}_s^i f_s\left(\bar{y}_0^k\right) \bar{\theta}_k^i$$

Then

$$\bar{x}_0^i - y_0^k \bar{\theta}_k^i + \sum_{s \in S^i} \bar{\pi}_s^i \left[f_s\left(y_0^k\right) \bar{\theta}_k^i + \bar{x}_s^i\right] > \bar{x}_0^i - \bar{y}_0^k \bar{\theta}_k^i + \sum_{s \in S^i} \bar{\pi}_s^i \left[f_s\left(\bar{y}_0^k\right) \bar{\theta}_k^i + \bar{x}_s^i\right]$$

By definition of $\bar{\pi}_s^i$, given assumption U, this is equivalent to

$$\begin{aligned}
&\left(\frac{\partial u^i}{\partial x_0^i}\left(\bar{x}^i\right)\right)\left(\bar{x}_0^i - y_0^k \bar{\theta}_k^i\right) + \sum_{s \in S^i}\left(\frac{\partial u^i}{\partial x^i(s)}\left(\bar{x}^i\right)\right)\left[f_s\left(y_0^k\right) \bar{\theta}_k^i + \bar{x}_s^i\right] \\
> &\left(\frac{\partial u^i}{\partial x_0^i}\left(\bar{x}^i\right)\right)\left(\bar{x}_0^i - \bar{y}_0^k \bar{\theta}_k^i\right) + \sum_{s \in S^i}\left(\frac{\partial u^i}{\partial x^i(s)}\left(\bar{x}^i\right)\right)\left[f_s\left(\bar{y}_0^k\right) \bar{\theta}_k^i + \bar{x}_s^i\right]
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left(\frac{\partial u^i}{\partial x_0^i}\left(\bar{x}^i\right)\right)\left(\bar{x}_0^i - \left(y_0^k - \bar{y}_0^k\right) \bar{\theta}_k^i\right) + \sum_{s \in S^i}\left(\frac{\partial u^i}{\partial x^i(s)}\left(\bar{x}^i\right)\right)\left[\left(f_s\left(y_0^k\right) - f_s\left(\bar{y}_0^k\right)\right) \bar{\theta}_k^i + \bar{x}_s^i\right] \\
> &\left(\frac{\partial u^i}{\partial x_0^i}\left(\bar{x}^i\right)\right) \bar{x}_0^i + \sum_{s \in S^i}\left(\frac{\partial u^i}{\partial x^i(s)}\left(\bar{x}^i\right)\right) \bar{x}_s^i
\end{aligned}$$

Applying proposition 31.2 (ii) in [5], pp. 359, there exists a scalar $\mu \in (0, 1]$ such that

$$\begin{aligned}
&u^i\left(\left(\bar{x}_0^i - \mu\left(y_0^k - \bar{y}_0^k\right) \bar{\theta}_k^i\right); \left(\mu\left(f_s\left(y_0^k\right) - f_s\left(\bar{y}_0^k\right)\right) \bar{\theta}_k^i + \bar{x}_s^i\right)_{s \in S}\right) \\
> \; &u^i\left(\bar{x}_0^i; \left(\bar{x}_s^i\right)_{s \in S}\right)
\end{aligned}$$

Defining $\widehat{y}_0^k \equiv \bar{y}_0^k + \mu\left(y_0^k - \bar{y}_0^k\right)$ we contradict the NPIC.

I show the reverse now. Assume that i holds but not $NPIC$. Hence there exists y_0^k distinct from $\bar{y}_0^k$ such that

$$\begin{aligned} & u^i\left(\left(\bar{x}_0^i - \left(y_0^k - \bar{y}_0^k\right)\bar{\theta}_k^i\right); \left(\left(f_s\left(y_0^k\right) - f_s\left(\bar{y}_0^k\right)\right)\bar{\theta}_k^i + \bar{x}_s^i\right)_{s\in S}\right) \\ > \; & u^i\left(\bar{x}_0^i; \left(\bar{x}_s^i\right)_{s\in S}\right) \end{aligned}$$

By assumption U, u^i is strictly quasiconcave. Hence for any $\mu \in (0,1]$

$$\begin{aligned} & u^i\left(\left(\bar{x}_0^i - \mu\left(y_0^k - \bar{y}_0^k\right)\bar{\theta}_k^i\right); \left(\mu\left(f_s\left(y_0^k\right) - f_s\left(\bar{y}_0^k\right)\right)\bar{\theta}_k^i + \bar{x}_s^i\right)_{s\in S}\right) \\ > \; & u^i\left(\bar{x}_0^i; \left(\bar{x}_s^i\right)_{s\in S}\right) \end{aligned}$$

Define $\widehat{y}_0^k \equiv \bar{y}_0^k + \mu\left(y_0^k - \bar{y}_0^k\right)$ as before. By Proposition 31.2 (i) in [5], pp. 359, the last inequality implies

$$\begin{aligned} & \left(\frac{\partial u^i}{\partial x_0^i}\left(\bar{x}^i\right)\right)\left(\bar{x}_0^i - \left(\widehat{y}_0^k - \bar{y}_0^k\right)\bar{\theta}_k^i\right) + \sum_{s\in S^i}\left(\frac{\partial u^i}{\partial x^i(s)}\left(\bar{x}^i\right)\right)\left[\left(f_s\left(\widehat{y}_0^k\right) - f_s\left(\bar{y}_0^k\right)\right)\bar{\theta}_k^i + \bar{x}_s^i\right] \\ > \; & \left(\frac{\partial u^i}{\partial x_0^i}\left(\bar{x}^i\right)\right)\bar{x}_0^i + \sum_{s\in S^i}\left(\frac{\partial u^i}{\partial x^i(s)}\left(\bar{x}^i\right)\right)\bar{x}_s^i \end{aligned}$$

By assumption U again this implies, after obvious manipulations,

$$\bar{x}_0^i - \widehat{y}_0^k\bar{\theta}_k^i + \sum_{s\in S^i}\bar{\pi}_s^i\left[f_s\left(\widehat{y}_0^k\right)\bar{\theta}_k^i + \bar{x}_s^i\right] > \bar{x}_0^i - \bar{y}_0^k\bar{\theta}_k^i + \sum_{s\in S^i}\bar{\pi}_s^i\left[f_s\left(\bar{y}_0^k\right)\bar{\theta}_k^i + \bar{x}_s^i\right]$$

which implies

$$-\widehat{y}_0^k + \sum_{s\in S^i}\bar{\pi}_s^i f_s\left(\widehat{y}_0^k\right) > -\bar{y}_0^k + \sum_{s\in S^i}\bar{\pi}_s^i f_s\left(\bar{y}_0^k\right)$$

contradicting i.

It is immediate that ii implies i. It remains to show the converse. To do this I use the modified version of the Minkowski-Farkas lemma (Lemma 10). The proof of this lemma is presented after finishing this demonstration. Assume the lemma is true. Define the set

$$\begin{aligned} Z_k \; & \equiv \; Z_k\left(\bar{y}_0^k, \left(\bar{\pi}^i\right)_{i\in I_k}\right) \\ & \equiv \; \left\{ y \in \Re_{++}^{S^*+1} : -y_0 + \sum_{s\in S^i}\bar{\pi}_s^i y_s > -\bar{y}_0^k + \sum_{s\in S^i}\bar{\pi}_s^i f_s^k\left(\bar{y}_0^k\right) \right\} \end{aligned}$$

Hence, condition i is equivalent to $Z_k \cap Y^k = \emptyset$, where

$$Y^k = \left\{ y \in \Re^{S^*+1} : y_s^k = f_s^k\left(y_0^k\right) \right\}$$

Note that Z_k is open. Since it is also a convex subset in $\Re^{S^*+1}$, then it is true that $riZ_k = intZ_k$ (see [11], pp. 44). In fact, we can also consider both the closure of Z_k and of Y^k. By the standard separation theorem (see [11], theorems 11.1 and 11.3), there is a non-zero vector $\widetilde{\pi}$ such that for any y in clZ_k, for any $y^k \in Y^k$, it is true that

$$\begin{aligned} \widetilde{\pi}_0 y_0^k + \sum_s \widetilde{\pi}_s y_s^k &\le \widetilde{\pi}_0 y_0 + \sum_s \widetilde{\pi}_s y_s \\ \forall y^k &\in Y^k \\ \forall y &\in clZ_k \end{aligned}$$

and, because $\widetilde{\pi} \neq 0$, we can assume without loss of generality that $\widetilde{\pi}_0 \neq 0$. Define $\bar{\pi}_0 \equiv -\widetilde{\pi}_0$ and $\bar{\pi}_s \equiv \widetilde{\pi}_s$. Then the last inequality is equivalent to:

$$-\bar{\pi}_0 y_0^k + \sum_{s \in S^*} \bar{\pi}_s y_s^k \le -\bar{\pi}_0 y_0 + \sum_{s \in S^*} \bar{\pi}_s y_s$$

Now, we know that $\bar{y}^k \in clZ_k$. Then the following must hold.

$$\begin{aligned} -\bar{\pi}_0 y_0^k + \sum_{s \in S^*} \bar{\pi}_s y_s^k &\le -\bar{\pi}_0 \bar{y}_0^k + \sum_{s \in S^*} \bar{\pi}_s \bar{y}_s, \quad \forall y^k \in Y^k \\ &\Leftrightarrow \\ \bar{\pi} &\in N_{Y^k}\left(\bar{y}^k\right) \end{aligned}$$

where

$$N_{Y^k}\left(\bar{y}^k\right) \equiv \left\{ \bar{\pi} \in \Re^{S^*+1} \mid \bar{\pi} \cdot \bar{y}^k \ge \bar{\pi} \cdot y^k, \ \forall y^k \in Y^k \right\}$$

This last condition is in fact the consequence of $Z_k \cap Y^k = \emptyset$. It demands that the vector that forms the hyperplane *maximizes* the net present value of the production plan $\bar{y}^k$.

It is also true that $\bar{y}^k \in Y^k$ (since it is an equilibrium production plan). Therefore

$$-\bar{\pi}_0 \bar{y}_0^k + \sum_{s \in S^*} \bar{\pi}_s \bar{y}_s^k \le -\bar{\pi}_0 y_0 + \sum_{s \in S^*} \bar{\pi}_s y_s$$

for all y in Z_k. Now, Z_k is defined by a system of inequalities. Hence this last condition is equivalent to the following.

$$\begin{aligned} -\left(y_0 - \bar{y}_0^k\right) + \sum_{s \in S^i} \bar{\pi}_s^i \left(y_s - \bar{y}_s^k\right) &\ge 0 \quad \forall i \in I_k \\ &\Downarrow \\ -\bar{\pi}_0 \left(y_0 - \bar{y}_0^k\right) + \sum_{s \in S^*} \bar{\pi}_s \left(y_s - \bar{y}_s^k\right) &\ge 0 \end{aligned}$$

By Lemma 10, there is a set of non-negative scalars $\left(\widehat{\alpha}^1, ..., \widehat{\alpha}^I\right)$ such that $\bar{\pi} = \sum_{i=1}^{I} \widehat{\alpha}_i \widehat{\pi}^i$, with $\widehat{\pi}^i$ being defined in the statement of Lemma 10. Let $\alpha_i \equiv \widehat{\alpha}_i \,/\, \bar{\theta}_i^k$. Therefore $\bar{\pi} = \sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \widehat{\pi}^i$. Since $\bar{\pi}$ is different than 0, it must be true that there is at least some i such that $\alpha_i \bar{\theta}_i^k > 0$. But $\bar{\pi} \in N_{Y^k}\left(\bar{y}^k\right)$. Then $\sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \widehat{\pi}^i \in N_{Y^k}\left(\bar{y}^k\right)$. This implies that, for all y^k in Y^k,

$$\sum_{s \in S^*} \sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \widehat{\pi}_s^i f_s^k\left(\bar{y}_0^k\right) - \sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \widehat{\pi}_0^i \bar{y}_0^k \geq \sum_{s \in S^*} \sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \widehat{\pi}_s^i y_s^k - \sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \widehat{\pi}_0^i y_0^k$$

which is equivalent to

$$\sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \left\{ \sum_{s \in S^i} f_s^k\left(\bar{y}_0^k\right) - \bar{y}_0^k \right\} \geq \sum_{i=1}^{I} \alpha_i \bar{\theta}_i^k \left\{ \sum_{s \in S^i} f_s^k\left(y_0^k\right) - y_0^k \right\}$$

(recall that, if $y^k \in Y^k$, then $y_s^k = f_s^k\left(y_0^k\right)$). But this is just condition ii. ■

Proof of Lemma 10. I show that, if the inequality $\sum_{s \in S^*} \bar{\pi}_s y_s - \bar{\pi}_0 y_0 \geq 0$ is a consequence of the system of inequalities $\sum_{s \in S^i} \bar{\pi}_s^i y_s - y_0 \geq 0$ then it must be the case that $\bar{\pi} = \sum_i \alpha^i \widehat{\pi}^i$, where $\widehat{\pi}^i$ is defined above. Suppose this last equality is not satisfied. Let

$$K\left(\left(\bar{\pi}^i\right)_{i \in I^k}\right) \equiv \left\{ \pi \in \Re^{S^*+1} \mid \pi = \sum_{i \in I_k} \alpha^i \widehat{\pi}^i; \alpha_i \geq 0 \right\}$$

Therefore, $\bar{\pi} \notin K$. Hence, by the Separation Theorem for convex sets, there exists $\bar{y} \in \Re^{S^*+1}$ such that $\bar{\pi} \cdot \bar{y} < \pi \cdot \bar{y}$ for every π in K. We know that $0 \in K$ (define $\alpha_i = 0$ for all i). Hence $\bar{\pi} \cdot \bar{y} < 0$. On the other hand, it the last inequality implies that $\sum_{s \in S^i} \bar{\pi}_s^i \bar{y}_s - \bar{y}_0 \geq 0$ for all i. Suppose the contrary: there is an i' such that $\sum_{s \in S^{i'}} \bar{\pi}_s^{i'} \bar{y}_s - \bar{y}_0 < 0$. However, the vector $\widetilde{\pi} \equiv \alpha_{i'} \bar{\pi}^{i'}$ is in K, for any $\alpha_{i'} > 0$. Hence $\alpha_{i'}\left(\sum_{s \in S^{i'}} \bar{\pi}_s^{i'} \bar{y}_s - \bar{y}_0\right) < 0$, and with $\alpha_{i'} \to \infty$ we obtain that $\alpha_{i'}\left(\sum_{s \in S^{i'}} \bar{\pi}_s^{i'} \bar{y}_s - \bar{y}_0\right) \to -\infty$, contradicting $\bar{\pi} \cdot \bar{y} < \pi \cdot \bar{y}$ for every π in K. Hence $\sum_{s \in S^i} \bar{\pi}_s^i \bar{y}_s - \bar{y}_0 \geq 0$ for all i. But we also had that $\bar{\pi} \cdot \bar{y} \equiv \sum_{s \in S^*} \bar{\pi}_s \bar{y}_s - \bar{\pi}_0 \bar{y}_0 < 0$. Then we have contradicted the assumption that $\sum_{s \in S^i} \bar{\pi}_s^i y_s - y_0 \geq 0$ is the consequence of $\sum_{s \in S^i} \bar{\pi}_s^i \bar{y}_s - \bar{y}_0 \geq 0$ for every i.

The converse is immediate. ■

Proof of Proposition 11. The proof is standard. First, with the technological assumption the date-0 budget constraint can be written as

$$x_0^i = \omega_0^i + \sum_{j=1}^{J} \left(\psi_j^i - \phi_j^i\right) q_j - \sum_{k=1}^{K} \tau_k^i$$

$$x^i(s) = \omega^i(s) + \sum_{j=1}^{J} R_j(s)\,\kappa_j(s)\,\phi_j^i - \sum_{j=1}^{J} R_j(s)\,\psi_j^i + \sum_{k=1}^{K} \rho(s)\,\tau_k^i, \quad s \in S^i$$

where $\tau_k^i \equiv \theta_k^i y_0^k$. Hence the choice variable for agent i is τ_k^i, since $y_0^k \equiv \sum_{i=1}^{K} \tau_k^i$

The first order conditions, (together with assumptions C, U, and the fact that $\bar{\psi}_j > 0$ for all j) determine uniquely the solution to i's problem. Then this uniquely determine $\bar{\tau}_k^i$, and then total output of firm k is just the sum $\sum_{i=1}^{K} \bar{\tau}_k^i$. Therefore there can be no disagreement in terms of the optimal scale of production. ∎