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Why the Rich get Richer and the Poor get Poorer

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Abstract

The usual procedure in the field of optimal growth theory consists in maximizing a (discounted or not) sum of instantaneous utilities, called welfare. Such an optimality criterion implies that preferences are independent over time.

Following in the tradition of Irwing Fisher, *Koopmans presented* an alternative for the case of discrete time periods; he used an assumption of limited non-complementarity over time, and showed that there exist welfare functions for which the rate of time preference is variable. Later he and others showed that the implications are that even in the simplest situations described by the neoclassical growth model initial conditions affect the long run optimal path.

Equivalent results for the case of continuous time have been reached by the present author.

A similar approach by Uzawa reaches different results due to his particular assumptions; his optimal paths are, in the long run, independent of initial wealth. Blanchard and Fischer have critizised

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Uzawa's increasing rate of time preference, which also is at variance with Irving Fisher's original treatment of the subject.

The particular case in which the resulting welfare function can be explicitly represented as an integral as in Uzawa's essay has been analyzed elsewhere by the present author, but is not covered by the other studies which assume that the welfare functional is quasi-concave. The results for growth theory obtained illustrate the use of such a welfare function taking into account Fisher's form for the pure rate of time preference; the qualitative behavior of optimal growth paths is there seen to be similar to that described previously, including the multiplicity of asymptotic growth paths, with long run situations depending on the initial endowments. Thus preferences may lead to a "poverty trap" even in the case of a well behaved neoclassical technology. Such cases can then be described as in the title of this essay, the rich becoming richer and the poor getting poorer.

The present essay presents more rigurous arguments and additional examples.

1 Introduction.

The usual procedure in the field of optimal growth theory consists in maximizing a (discounted or not) sum of instantaneous utilities, called welfare. Such an optimality criterion implies that preferences are independent over time.

Following in the tradition of Irwing Fisher, Koopmans [1960] presented an alternative for the case of discrete time periods; he used an assumption of limited non-complementarity over time, and showed that there exist welfare functions for which the rate of time preference is variable. In a later study with Beals (Beals and Koopmans [1969]; see also Iwai [1972]) he showed that the implications are that even in the simplest situations described by the neoclassical growth model initial conditions affect the long run optimal path.

Equivalent results for the case of continuous time have been reached by the present author (Mantel [1966], [1967a], [1967b], [1993]).

A similar approach by Uzawa [1968] reaches different results due to his particular assumptions; his optimal paths are, in the long run, independent of initial wealth. Blanchard and Fischer [1991], referring to Uzawa's increasing rate of time preference, state that this "is not particularly attractive as a description of preferences and is not recommended for general use". Irving Fisher, the father of the creature, explains in his *Theory of Interest* [1930], pg 247 that "near the minimum of subsistence ... to give up one iota of this year's income in exchange for any amount promised for next year would mean too great a privation in the present. ...his rate of time preference will gradually decrease ... that is, the larger the income, other things remaining the same, the smaller the degree of impatience."

The particular case in which the resulting welfare function can be explicitly represented as an integral as in Uzawa's essay has been analyzed elsewhere (Mantel [1967c]), but is not covered by the other studies which assume that the welfare functional is quasi-concave. The results for growth theory obtained illustrate the use of such a welfare function taking into account Fisher's form for the pure rate of time preference; the qualitative behavior of optimal growth paths is there seen to be similar to that described previously, including the multiplicity of asymptotic growth paths, with long run situations depending on the initial endowments. Thus preferences may lead to a "poverty trap" even in the case of a well behaved neoclassical technology. In such cases the rich desire to become richer, whereas the poor prefer getting poorer; under high levels of initial capital stock society may wish to save and accumulate more, while the same preferences may lead to dissaving if initial wealth is below some critical level.

The present essay presents more rigorous arguments and additional examples.

In the main text only the results will be given, with some indication as to their proofs. Detailed proofs are left for the Appendix.

2 Preference over time.

The present section presents briefly some of the results needed in the sequel. A more thorough analysis has been carried out previously (Mantel [1966], [1967a], [1970], [1993]), where the gap between the two approaches — continuous vs. discrete time— has been bridged, by showing that a suitable limiting process allows one to define a utility function for continuous time with a variable rate of time preference. The main result is that the assumptions of stationarity and limited non-complementarity over time imply that the prospective utility of a consumption program extending from the present to the unlimited future, can be evaluated as the initial value of the solution of a differential equation, relating the marginal increase in prospective utility due to the advancing of the program, to the level of that utility and to the instantaneous utility of the commodity bundle thereby discarded. Nevertheless some of the proofs are not appropriate for the present case, since here the welfare functional may not be quasi-concave.

A *time-path* or *program* is a real-valued function $z(t)$, where the non-negative real argument t represents time. The present moment is $t = 0$, and the planning horizon of the family or society extends to the infinite future. Admissible functions are bounded and piece-wise continuous. The set of all admissible paths will be called Z.

A *consumption path* $x(t) \in Z$ is an instance of an admissible path. The set of admissible consumption programs X consists of those admissible paths for which the consumption rate is never negative, so that $x(t) \geq 0$ for all t. A *welfare function —prospective utility* in Koopmans' terminology— is a real valued function W defined on the set X of consumption programs. The *immediate* or *instantaneous utility* of a consumption rate x is the value of the real valued function $u(x)$. It should be noted that this definition is at variance with Koopmans' concept adhered to in Mantel [1993]. In the present essay, the relation between these two concepts is given by the integral

$$W[{}_0x] = \int_0^\infty e^{-\int_0^s \rho[x(v)]dv} u[x(s)]\, ds,$$

where the real-valued function $\rho(x)$ is the (psychological) *rate of time preference,* and ${}_0x$ stands for the program initiating at the present time 0.

The welfare function satisfies the following postulates, originally stated by Koopmans for discrete time. For the continuous time case, see the author's essays already cited. Here only verbal statements will be provided.

P1. (Sensitivity). There exist two admissible programs which agree with each other from some time on with different welfare levels.

This postulate serves the purpose of excluding the uninteresting case in which all consumption programs are equivalent to each other, which then trivially would all be optimal.

P2. *(Limited non-complementarity over time).* The ordering of two initially constant programs with the same tail —*i.e.* which coincide after a certain moment— is not affected if their common tail is replaced by another one, as long as after the replacement both programs still have equal ending sections.

The limited non-complementarity postulate is the central assumption which allows writing the welfare function in terms of a differential equation.

The condition that the comparison be limited to programs which are initially constant is essential; without it, the present and the next postulates would imply *that the utility function can be taken to be additive*, expressed as an integral of instantaneous utilities, discounted at a constant rate. This assertion has been proved before; the proof will not be repeated.

P3. (Stationarity). The ordering of two programs which coincide initially for some time is the same one obtains by discarding the common initial period and advancing these programs for a time duration equal to that period —the ordering of the tails—.

The purpose of this postulate is not its realism; one might argue that future generations have different tastes, so that the evaluation of a program from their perspective is not equal to the present generation's evaluation of the same program from today's perspective if it were to start today. The reason for requiring this postulate to be satisfied is to isolate the pure time preference effect from changes in tastes, in the belief that given sufficient freedom in the choice of preferences any development path may be justified. This would then provide no proof that development paths behave differently in the long run solely on the grounds of different initial endowments in response to a variable rate of time preference.

P4. (Extreme programs). There exist a best and a worst program, with finite welfare levels.

Thus the welfare of an admissible consumption program is bounded.

It has been shown (Mantel [1967a], [1970]) that under suitable continuity assumptions these postulates imply the existence of an aggregator function whose arguments are the rate of consumption x and the welfare level W which is strictly decreasing in its first argument, and —if the representation of preferences is chosen appropriately— is strictly increasing in its second argument.

The aggregator function has the property that the welfare of a program can be evaluated by solving the following differential equation with bounded end condition for its initial value. The solution is given by a *welfare path* $W(t)$ such that $W(t)$ is the prospective utility one would derive from implementing today the tail of the program intended to start at time t.

In the present case, this means that

$$W(t) \equiv W[{}_t x] = \int_t^{\infty} e^{-\int_t^s \rho[x(v)]\,dv} u[x(s)]\, ds.$$

By differentiation it is easily checked that W satisfies the differential equation

$$\dot{W}(t) = \rho[x(t)]\, W(t) - u[x(t)] \tag{1}$$

The interpretation of differential equation (1) is as follows. The prospective utility of the consumption program starting at time t is $W(t)$. The program offers a consumption rate $x(t)$ at that time. The aggregator function —the right hand side of the differential equation— uses this information to indicate that if those two quantities are known, advancing the program by discarding the consumption of the first instants after the current time t achieves an increase in prospective utility at the rate $\dot{W}(t)$.

For the purposes of the maximization of welfare to be carried out in the next section, it will be assumed that the following conditions hold for the instantaneous utility function $u()$ and the rate of time preference function $\rho()$.

P5. (Utility aggregator). The *utility-aggregator* function $\rho[x]\,W - u[x]$ satisfies

1. $u(x)$ and $\rho(x)$ are continuous and twice continuously differentiable for $x \geq 0$,
2. $u()$ is strictly concave, $\rho()$ is strictly convex,
3. $u'(x) > 0; \rho'(x) < 0$, for all $x \geq 0$, and $u'(0) = +\infty$, $u(0) \geq 0$.
4. $\rho(x) \geq \epsilon > 0$ for some constant ϵ, for all x.

It is easily verified that such an aggregator function produces a welfare function which satisfies the postulates. The level curves of a function satisfying Postulate P5 are shown in Figure 1a, derived from the utility function shown in Figure 1b and the rate of time preference function in Figure 1c.

| Figure 1a about here |

| Figure 1b about here |

Figure 1c about here

For uniformly bounded admissible consumption programs ${}_0x$ one has

$$W({}_0x) = \lim_{T \to \infty} W(0; T, W(T))$$

where $W(t; T, W(T))$ *is a solution of the differential equation* (1) *with any* end condition satisfying

$$W(T; T, W(T)) = W(T)$$

with

$$0 \leq \underline{W} \leq W(T) \leq \bar{W}.$$

We shall give $\rho(.)$ the name of *instantaneous rate of time preference.* As *will be seen it acts as a discount rate. Note that it is* independent of the representation of preferences only for constant programs; in the general case its value depends on the (welfare) utility scale. In the present situation, it coincides with the concept of a *pure rate of time preference* used elsewhere; in more general situations the two concepts are equal only in the case of stationary programs (see Mantel [1993]).

3 The technology and feasibility.

The technology —here we draw heavily on previous work, since there is no innovation offered— will be described by a simple neoclassical aggregate production function with the following properties.

P6. (Technology). The real-valued production function $f(k)$ —where the non-negative real number k denotes capital per capita— is

1. continuous, twice continuously differentiable for $k > 0$,
2. $f(0) = 0$; $f'(0) > 0$; $f''(k) < 0$.
3. There exists a $k_m > 0$ such that $f(k_m) = 0$.

Here it is assumed that there exists only one good, used both for consumption and for accumulation. The symbol k stands for the *capital-labor ratio,* $f(.)$ for the *output-labor ratio* —the latter net of maintenance and *other costs, including the investment necessary for keeping the capital-labor* ratio constant—. The second assumption is standard, and states that capital is an indispensable input and that output per capita is an initially increasing and concave function of capital per capita. The last line can be justified in an economy *with a growing labor force, where it is conceivable that* as labor becomes scarce it will be impossible to produce enough to sustain the capital-labor ratio. In the sequel no reference will be made to the rate of growth of labor, which will be assumed to be constant. All relevant variables will be expressed in *per capita* terms.

Figure 2 shows the graph of a function satisfying Postulate P6 on the production function.

Figure 2 about here

Denote the highest sustainable —"golden rule"— consumption rate by $\bar{x}$, the corresponding level of capital by $\bar{k}$, so that both quantities are positive and $f'\left(\bar{k}\right) = 0$; $\bar{x} = f\left(\bar{k}\right)$.

A *capital path* is an admissible path ${}_0k$; it is *feasible for an initial capital stock* k if $k(0) = k$ and $0 \leq s \leq t$ implies

$$k(s)\, e^{-\delta(t-s)} \leq k(t) \leq k(s) + \int_s^t f(k(v))\, dv,$$

where $0 < \delta < \infty$ represents the rate of capital deterioration —depreciation plus the growth rate of labor—, the highest rate at which capital can be used up. Thus a feasible capital path is differentiable almost everywhere in the sense of Lebesgue and satisfies the corresponding differential inequalities

$$-\delta k(t) \leq \dot{k}(t) \leq f(k(t)).$$

The associated consumption path ${}_0x$ satisfies

$$x(t) = f(k(t)) - \dot{k}(t), \tag{2}$$

so that $0 \leq x(t) \leq f(k(t)) + \delta k(t)$.

To simplify the exposition, the analysis will be restricted to those situations in which the initial capital stock is productive, *i.e.* $0 < k(0) < k_m$. In that case feasibility implies $0 < k(t) < k_m$ for all t. Consequently the capital path is uniformly bounded, and so is the consumption path, with

$0 \leq x(t) \leq x^* \equiv f(k^*) + \delta k^* \equiv \max\{f(k) + \delta k | 0 \leq k \leq k_m\}$. The problem to be solved now consists in determining the optimal feasible capital, consumption and welfare programs.

The analysis will be simplified by decomposing the maximization process into several elementary steps. With any feasible program one associates certain tentative implicit prices for the consumption good and the use of the same as capital good; these prices can then be used to compare different programs. In the end, for the optimal program, they turn out to equal the dual or co-state variables of the maximization problem.

Define the (psychological) *discount factor*, λ, and the *prices*, p, q, *associated* with a feasible path $({}_0W, {}_0x, {}_0k)$ as follows. The discount factor is

$$\lambda(t) \equiv e^{-\int_0^t \rho(x(s))ds} \tag{3}$$

and satisfies the inequalities

$$e^{-\rho(0)t} \leq \lambda(t) \leq e^{-\rho(x^*)t} < e^{-\epsilon t} < 1$$

for all t, due to **P5**.

This expression uses the instantaneous rate of time preference ρ as a discount rate to evaluate the relative merit of events at time t as seen from the present time 0.

For the price of the consumption good at time t, take the discounted increase in welfare due to a marginal increase in consumption, *i.e.*

$$p(t) \equiv \lambda(t)(u'[x(t)] - W(t)\rho'[x(t)]). \tag{4}$$

Thus $p(t) \leq \lambda(t)\, u'(x(t)) \leq u'(x(t)) \leq u'(a)$ if $a \leq x(t)$.

For the rental price of the use of capital take the value of its marginal product at consumption prices,

$$q(t) \equiv p(t) f'(k(t)) \tag{5}$$

These definitions allow the following results to be obtained.

Proposition 1. If $({}_0W, {}_0x, {}_0k)$ and $\left({}_0\hat{W}, {}_0\hat{x}, {}_0\hat{k}\right)$ are feasible, then, if $\xi(t) \equiv x(t) - \hat{x}(t)$,

$$W({}_0x) - W({}_0\hat{x}) \leq \int_0^\infty \hat{p}(t)\xi(t)\,dt + \int_0^\infty \left[\frac{\lambda(t)}{\hat{\lambda}(t)} - 1\right]\hat{p}(t)\xi(t)\,dt$$

This proposition —a result similar, though weaker, to Koopmans' [1965] proposition (F) for a constant rate of time preference— states that the difference between the welfare levels or prospective utilities of two consumption paths —the left hand side of the inequality— does not exceed the present or *discounted value of the difference of the two consumption streams* —the right hand side— by much, where these two infinite consumption programs are evaluated at the discounted prices of the consumption good associated with the second path. The second integral in this expression is of negligible *importance in program changes of short duration, and can be neglected if* one seeks necessary conditions. A rigurous proof is in the appendix.

The next proposition compares the consumption programs with the corresponding capital programs.

Proposition 2. *If* $({}_0W, {}_0x, {}_0k)$ and $\left({}_0\hat{W}, {}_0\hat{x}, {}_0\hat{k}\right)$ are feasible with the same initial capital, then

$$\int_0^\infty \hat{p}(t)(x(t)-\hat{x}(t))\,dt \leq \int_0^\infty \left(\hat{q}(t)+\dot{\hat{p}}(t)\right)\left(k(t)-\hat{k}(t)\right)dt + \lim_{t\to\infty} \hat{p}(t)\hat{k}(t)$$

This proposition —comparable to Koopmans' [1965] proposition (G) for a constant rate of *time preference— states that, evaluated at the implicit* prices of the second path, the present (discounted) value of the difference of the two consumption paths —the left hand side of the inequality— does not exceed the difference in the present value of the two capital services (evaluated at the price for the use of *capital services* $\hat{q}$) *plus capital gains* (due to changes in the price of the assets $\dot{\hat{p}}$) —these two concepts are represented by the terms under the integral sign on the right hand side—, plus the scrap value of the final capital stock of the second path —the last term, the limit of the value of the capital stock as time tends to infinity—.

4 Optimality.

This section follows as far as possible and almost verbatim the corresponding section in Mantel [1993] so that the two cases can be easily compared. The main difference resides in the almost imperceptible omission of the word "sufficient" from proposition 3, which of course is crucial as always when not all *convexity requirements are met.* [1]

[1] The other major difference is that $\rho()$ now replaces both $F_W()$ and $r()$.

The two propositions of the previous section lead immediately to the conditions that must be satisfied by optimal programs. Linking the two inequalities in propositions 1 and 2 together —the right hand side of the first is as close to the left hand side of the second for sufficiently short time durations— *the necessity of the condition in the next proposition follows* from the maximum principle of optimal control theory. A more intuitive argument is given below.

Proposition 3. If the rate of capital deterioration $\delta$is sufficiently large, necessary for the optimality of the given path $\left({}_0\hat{W},{}_0\hat{x},{}_0\hat{k}\right)$is that its implicit prices satisfy

$$\hat{q}(t) + \dot{\hat{p}}(t) = 0 \qquad \text{for} \quad t \geq 0 \tag{6}$$

and that the transversality condition

$$\lim_{t\to\infty} \hat{p}(t)\,\hat{k}(t) = 0$$

hold.

Equation (6) can be rephrased as saying that the discounted price of the consumption good should fall at a rate equal to the rental price of the capital services it provides. Note that this result is in line with the necessity of Koopmans' [1965] proposition (H) for a constant rate of time preference.

A *heuristic argument, similar to the Keynes-Ramsey-Koopmans argu*ment —first presented by Ramsey [1928], who attributes it to Keynes for the case of a zero rate of time preference, and later by Koopmans [1965] for a constant rate of time preference— is as follows. At any time t, increasing *consumption by a fraction ϵ of the investment rate, $\dot{k}$, during a short time* interval η means an increase in consumption of $\Delta x = \epsilon\dot{k}$. This produces a gain in welfare equal to

$$\begin{aligned}
\Delta W &= \Delta\left(W\left({}_0x\right) - W\left({}_\eta x\right)\right) \\
&= -\eta\Delta\dot{W} = -\eta\Delta\left[\rho W - u\right] \\
&= -\eta\left[\rho' W - u'\right]\Delta x = \eta\epsilon\left[u' - \rho' W\right]
\end{aligned}$$

and a loss —due to postponement of capital accumulation by a fraction ϵ of the time period η— equal to $\epsilon\eta\dot{W}$. The net gain is therefore

$$-\left(\left[\rho' W - u'\right]\dot{k} + \dot{W}\right)\eta\epsilon$$

and should not be positive if the capital path is to be optimal. Since ϵ can have any sign, it follows that

$$\dot{W} + [\rho' W - u'] \dot{k} = 0 \tag{7}$$

The foregoing argument can be shortened considerably, and perhaps made more intuitive, if one chooses the time unit to correspond to a very short interval, say a second or a fraction thereof. One can then increase the consumption rate during the second beginning at time t by cutting investment to zero, thereby *earning a welfare benefit of* $-[\rho' W - u'] \dot{k}$. The new capital stock will now be reached a second later, so that the consumption program will have to be postponed by a second at a welfare cost given by $\dot{W}$. At an optimum the net benefit is zero, so that equation (7) is again satisfied.

Multiplying this equation by the discount factor λ *and using the definition* (4) of the price p one the obtains $-\lambda \dot{W} + p\dot{k} = 0$, or replacing the time-derivatives from equations (1) and (2), $-\lambda(\rho W - u) + p(f(k) - x) = 0$. Computing the derivative with respect to time t of this identity and reordering the terms gives

$$-(\lambda[\rho' W - u'] + p)\dot{x} - \left(\lambda\rho + \dot{\lambda}\right)\dot{W} + (q + \dot{p})\dot{k} = 0.$$

The first two terms drop out because of the definitions of p in (4) and λ in (3). Thus if the investment rate is not zero, the equality (6) follows.

Note that the zero net welfare benefit condition can be written as

$$p/\lambda = u' - \rho' W = \dot{W}/\dot{k} = \frac{dW}{dk}$$

which shows that the undiscounted price of the consumption good measures the welfare effect of a marginal addition to the capital stock.

Proposition 4. *For any initial capital stock* $0 \leq k(0) < k_m$ *there exists* an optimal path.

The purpose of this result is to confirm that one is not making statements about non-existing items.

Proposition 5. *The welfare levels of optimal programs are an increasing* function of the initial capital k.

That is to say that $W({}_0x)$ *increases with* $k(0)$. *This confirms the intuition* that more resources are better.

Define a capital path to be strictly monotone if it is constant or either always strictly increasing or else always strictly decreasing. Then one has

Proposition 6. Optimal capital paths are strictly monotone.

In other words, under the present assumptions, optimality excludes bulges or cycles in capital programs. *In this the present analysis does not differ* qualitatively from the standard result obtained with a constant rate of time preference.

Proposition 7. Optimal capital paths are strictly increasing (decreasing, constant) if the marginal product of the initial capital stock exceeds (is less than, equals) the pure rate of time preference corresponding to a constant *capital path equal to that initial capital stock, that is, if*

$$f'(k) > (<, =) \ \rho[f(k)]. \tag{8}$$

This result is also true if the rate of time preference is constant. The difference resides in that if the pure rate of time preference is constant then there is only one capital-labor ratio with a marginal product equal to it, whereas if it is decreasing there may be several solutions to the equality in relations (8). This central result of the present investigation is summarized in the next proposition.

Figure 3 graphs the marginal product of capital and the pure rate of time preference corresponding to stationary programs. As drawn they cross at two points giving rise to two stationary solutions, not counting the origin. One situation is stable, the other two (one of them the origin) are unstable.

Figure 3 about here

Proposition 8. If the initial capital stock is very large, the optimal path will be strictly decreasing. If $\rho(0) < f'(0)$ and the initial capital stock is very *low the path will be strictly increasing, else it will decrease toward zero.* For intermediate initial capital stocks, there may be several intervals for which the path rises or for which it falls, separated by constant paths along which the pure rate of time preference equals the marginal product of capital.

Figure 4 shows the capital paths corresponding to the marginal product of capital and the rate of time preference schedules of Figure 3. The monotonicity property of Proposition 6 is illustrated, and it can be seen how the constant equilibrium paths separate those that are always increasing or always decreasing.

Figure 4 about here

5 An example.

In the present section an explicit example will be provided. The solutions have been computed, in order to show that the foregoing results are indeed consistent.

Let

$$\rho(x) = \epsilon + \frac{\beta}{\alpha + \gamma \cdot x + \delta \cdot x^2}$$

$$f(k = c \cdot k + b \cdot log(1 + \frac{k}{a}) - \left(\frac{1}{3}\right) \cdot d \cdot k^3$$

$$u(x) = \frac{1}{1-\eta} x^{1-\eta}$$

Then the following values for the parameters produce Figure 5,

$$(\alpha, \beta, \gamma, \delta, \epsilon, \eta) = (2.5, .3, 5, 10, .04, .5)$$
$$(a, b, c, d) = (.25, .03, .08, .0001),$$

if the time unit is taken to be equal to 5, providing the system of differential equations to be *satisfied by an optimal path*,

$$\begin{pmatrix} \dot{k} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} f(k) - x \\ \eta \cdot x \cdot [f'(k) - \rho(x) - \rho'(x)(f(k) - x)] \end{pmatrix}$$

Figure 5 about here

The figure shows the phase plane for the two variables x and k.

The isoclines are the dotted lines. It should be noted that the curve $\dot{k} = 0$ coincides with the production function; the other lines correspond to $\dot{x} = 0$ —one component is the closed curve near the origin, the other the nearly vertical curve cutting the picture in half. There are three intersections —as *compared to the two intersections shown in the previous sections— plus the* origin which locate the stationary solutions.

Several trajectories have been drawn, for different pairs of initial conditions, given by the pairs

$$\begin{pmatrix} k_0 \\ x_0 \end{pmatrix} = \left\{ \begin{array}{c} \begin{pmatrix} 2 \\ .2 \end{pmatrix}, \begin{pmatrix} 4 \\ .4 \end{pmatrix}, \begin{pmatrix} 4 \\ .43 \end{pmatrix}, \begin{pmatrix} 4 \\ .45 \end{pmatrix}, \begin{pmatrix} 6 \\ .5 \end{pmatrix}, \\ \cdots \begin{pmatrix} 6.1 \\ .5025 \end{pmatrix}, \begin{pmatrix} 6.2 \\ .505 \end{pmatrix}, \begin{pmatrix} 6.4 \\ .51 \end{pmatrix}, \begin{pmatrix} 8 \\ .55 \end{pmatrix}, \begin{pmatrix} 10 \\ .6 \end{pmatrix}, \\ \cdots \begin{pmatrix} 22 \\ 2 \end{pmatrix}, \begin{pmatrix} 24 \\ 2 \end{pmatrix}, \begin{pmatrix} 25 \\ 2 \end{pmatrix}, \begin{pmatrix} 30 \\ 2 \end{pmatrix} \end{array} \right\}.$$

All correspond to the same time duration. It is clear that only a few are near *the optimal solutions —not drawn— converging to the two stable points.*

6 Conclusion.

The *present investigation started with setting* out *a welfare function* for a family or a social planner wishing to design an optimal growth program in a neoclassical setting. "The proof of the cake is in the eating", which in the case of an economist in the position to advise the planner means that it is desirable to try out *several criteria for optimal growth* so as to ascertain the effects these have on the shape of the resulting optimal programs. It is difficult to ask the planners for their preferences, so it will be simpler to deduce them from their choice among optimal paths obtained from different optimality criteria.

A welfare function has been presented which is not so simple as to reduce to one with a *constant pure rate of time preference, but still simple* enough to be amenable to analysis, using the large body of results pertaining to optimal control theory.

The results that have been obtained show that on the one hand there are *similarities with the case of a constant rate of time preference,* in that the capital paths are one of three types,

1. constant for all time, in case that initially the pure rate of time preference coincides with the marginal productivity of capital;

2. strictly increasing, accumulating capital by consuming less than is produced, approaching a long run capital-labor ratio asymptotically in case the pure rate of time preference falls initially short of the marginal productivity of capital;

3. *strictly decreasing, decumulating capital by consuming more than is* produced, again approaching a long run capital-labor ratio asymptotically, in case the pure rate of time preference exceeds initially the marginal productivity of capital.

On the other hand there are important differences.

1. In the case of a constant —or increasing, as proposed by Uzawa— rate of time preference there exists a unique capital-labor ratio to which all *capital programs tend in the long run independently of the initial* endowment of the economy. In other words, poor societies will restrict their consumption to accumulate capital until the long run capital-labor ratio is reached, whereas rich societies will eat up their capital *until that same long run capital-labor ratio is attained.*

2. In the case of a variable rate of time preference —if it is falling as *proposed by Irving Fisher—, on the other hand, there may exist* a multiplicity of long run relative endowments. This means that the development path of an economy depends on its initial endowments; society is not willing to disregard its past.

It seems quite reasonable to expect to find situations in which there are at least two different capital-labor ratios at which the pure rate of time preference equals *the marginal product of capital.* In such a case, a very poor society may decide that the effort to accumulate capital is too high, that the benefits will take too long to be reaped, and thus embark in a high consumption program leading to a low —perhaps zero— long run capital-labor ratio. *On the other hand*, a somewhat richer society with an initial capital endowment exceeding some critical amount, may have sufficient incentives to decide to undertake the effort, to tighten their belts by consuming less, to accumulate and reach a long run capital-labor ratio that is higher than the present one.

More than two coincidences between the pure rate of time preference and the marginal product of capital are possible —as shown by the example— but do not seem to be plausible.

When the rate of time preference is allowed to vary, a country may decide not to undertake the effort of economic development when its initial capital endowment is below some critical level, whereas if it were above that level it would be willing to sacrifice its present generation for the well-being of the future ones. It is impossible to obtain such a result with a constant or *increasing rate of time preference in the case of a simple neoclassical* technology.

7 APPENDIX

7.1 The problem

$$\begin{gathered}\max \int_0^\infty e^{-\int_0^t \rho(x)\,ds} u(x)\; dt \\ \text{s.t. } \dot{k} = f(k) - x; k(0) = k_0 \\ x \geq 0; k + \delta k \geq 0.\end{gathered}$$

where

$$\begin{gathered}u' \geq 0; u'' \leq 0; u(0) \geq 0 \\ \rho' < 0; \rho'' > 0; \rho \geq \epsilon > 0 \\ f(0) = 0; f'(0) > 0; f'' < 0; \left(\exists \hat{k} > 0\right)\; f'\left(\hat{k}\right) = 0 \\ \delta > 0 \\ k_0 > 0\end{gathered}$$

7.2 Existence (proof of proposition 4)

Equivalent problem is

$$\begin{gathered}\max \int_0^\infty \lambda\, u(x)\; dt \\ \text{s.t. } \dot{k} = f(k) - x; k(0) = k_0 > 0 \\ \dot{\lambda} = -\rho(x)\,\lambda;\; \lambda(0) = 1 \\ x \geq 0; \dot{k} + \delta k \geq 0.\end{gathered}$$

Our assumptions guarantee that Theorem 4, pg. 259, in Lee and Markus [1967] *on the existence of optimal controls with magnitude constraints*, is applicable.

To see this, consider the auxiliary problem in which the differential equation for λ is relaxed to the differential inequality $\dot{\lambda} \leq -\rho(x)\,\lambda$. This problem has a convex, compact velocity set

$$V(\lambda, k) \equiv \left\{ \left(\begin{array}{c} \dot{\lambda} \\ \dot{k} \end{array} \right) = \left(\begin{array}{c} -\rho(x)\,\lambda - z \\ f(k) - x \end{array} \right) \middle| 0 \leq x \leq f(k) + \delta k; z \geq 0 \right\},$$

hence there exists an optimal control for this auxiliary problem. But in the optimum the differential inequality becomes an equation since $\dot{\mu} = -\rho\mu; \mu(0) = 1$ implies that for all t

$$\ln(\lambda) \leq -\int_0^t \rho\, dt = \ln(\mu).$$

Because of the assumption $u(.) \geq 0$, one can substitute μ for λ. □

Define

$$W(t) \equiv \frac{1}{\lambda(t)} \int_t^\infty \lambda(s)\, u(x(s))\, ds.$$

Since $x(.)$ is bounded, $u(.)$ is continuous, and $\lambda(s) < \lambda(t)$ when $s > t$, *this definition implies that $W(.)$ is bounded. Consequently, since $\lambda \to 0$ as* $t \to \infty$, one has $\lim_{t\to\infty} \lambda(t) W(t) = 0$

Differentiating the definition of $W(.)$ gives the differential equation

$$\dot{W} = -\frac{\lambda u}{\lambda} - W\frac{\dot{\lambda}}{\lambda} = W\rho - u; \lim_{t\to\infty} \lambda(t)\, W(t) = 0.$$

7.3 Necessary conditions for an optimum (proof of proposition 3)

Write the problem as follows in order to applay Pontryagin's Maximum Principle in the case of a compact, constant control set U,

$$\begin{array}{c} \max \int_0^\infty \lambda\, u(\gamma h(k))\, dt \\ \text{s.t. } \dot{k} = f(k) - \gamma h(k); k(0) = k_0 > 0 \\ \dot{\lambda} = -\rho(\gamma h(k))\, \lambda;\ \lambda(0) = 1 \\ \gamma \in U \equiv [0, 1], \end{array}$$

where $h(k) \equiv f(k) + \delta k$. The Hamiltonian is

$$H \equiv \lambda u(\gamma h(k)) - W\lambda\rho(\gamma h(k)) + p[f(k) - \gamma h(k)],$$

where W, p are the costate variables. Thus the following necessary conditions can be derived.

$$\begin{array}{ll} \gamma \in \arg\max_{\gamma\in U} & \{H\}\ ; H = 0 \\ H_\lambda & = -\dot{W} \\ H_k & = -\dot{p} \end{array}$$

together with the transversality conditions

$$\lim_{t\to\infty} W\lambda = \lim_{t\to\infty} pk = 0,$$

since the enpoints are free.

The *first condition* gives the *further* necessary conditions

$$[\lambda(u' - W\rho') - p]\left\{\begin{array}{c}\leq\\=\\\geq\end{array}\right\} 0 \text{ if } \gamma \left\{\begin{array}{c}=0\\ \in)0,1(\\ =1\end{array}\right.$$

whereas the other two provide the differential equations

$$\begin{aligned}\dot{W} &= \rho W - u\\ \dot{p} &= -\{[\lambda u' - W\lambda\rho' - p]\gamma h' + pf'\}\left\{\begin{array}{l}= -pf' \text{ if } \gamma < 1\\ \leq -pf' \text{ if } \gamma = 1\end{array}\right.\end{aligned}$$

Furthermore, $H = 0$ implies

$$\lambda\dot{W} = p\dot{k}$$

for all t.

Now consider the case $\gamma = 0$. Then $\lambda(u'(0) - W\rho'(0)) \leq p$, which is impossible since $u'(0) = +\infty$. On the other hand, $\gamma = 1$ means $\dot{k} = -\delta k < 0$, hence $\dot{W} = \rho W - u < 0$ and $W < u[h(k)]/\rho[h(k)]$, situation which eventually must stop. In other words, if δ is suffieciently large as compared to k_o it will not arise. Consequently set $x(t) \equiv \gamma(t)h[k(t)]$ and consider the equality due to $0 < \gamma(t) < 1$,

$$u' - W\rho' = p/\lambda = \dot{W}/\dot{k} = (W\rho - u)/(f(k) - x),$$

so that one finally obtains the relation

$$u(x) - W\rho(x) + (u'(x) - W\rho'(x))(f(k) - x) = 0$$

(*)

Now consider the product λW. Since $\lambda W \to \infty$,

$$\lambda W = -\lambda W|_t^\infty = \int_t^\infty [\rho\lambda W - \lambda(\rho W - u)]\,dt = \int_t^\infty \lambda u\,dt,$$

which coincides with our previous definition of W. In particular, since $\lambda(0) = 1$,

$$W_o = \int_o^\infty \lambda u\,dt$$

provides the optimal value of the objective function. □

7.4 Proof of proposition 1

As in the previous demonstration, using the definitions,

$$\begin{aligned} W_o - \hat{W}_o &= -\lambda\left(W - \hat{W}\right)\Big|_o^\infty \\ &= \int_o^\infty \left[\rho\lambda\left(W - \hat{W}\right) - \lambda\left(\rho W - u - \hat{\rho}\hat{W} + \hat{u}\right)\right] dt \\ &= \int_o^\infty \lambda\left[u - \hat{W}\rho - \left(\hat{u} - \hat{W}\hat{\rho}\right)\right] dt \\ &\leq \int_o^\infty \lambda\left(\hat{u}' - \hat{\rho}'\hat{W}\right)(x - \hat{x})\, dt \\ &= \int_o^\infty \frac{\lambda\hat{p}}{\hat{\lambda}}(x - \hat{x})\, dt \end{aligned}$$

□

7.5 Proof of proposition 2

$$\begin{aligned} \int_o^\infty \hat{p}(x - \hat{x})\, dt &= \int_o^\infty \hat{p}\left[f - \hat{f} - \frac{d\left(k - \hat{k}\right)}{dt}\right] dt \\ &\leq \int_o^\infty \left(\hat{p}\hat{f}' + \hat{\dot{p}}\right)\left(k - \hat{k}\right) dt - \lim_{t\to\infty} \hat{p}\left(k - \hat{k}\right) \\ &\leq \int_o^\infty \left(\hat{q} + \hat{\dot{p}}\right)\left(k - \hat{k}\right) dt + \lim_{t\to\infty} \hat{p}\hat{k} \end{aligned}$$

□

7.6 Welfare levels increase with initial capital stock (proof of proposition 5)

Let $0 < k_0 < k_0^* < \bar{k} \in f^{-1}(\mathbb{R}_+) \cup h'^{-1}(\mathbb{R}_+)$. Let $k^*(t) = k_0^* e^{-\delta t}$ for $t \in [0, T]$, where $T \equiv \sup\{t | k^*(s) > k(s) \text{ for } s \in [0, t]\}$ —note that $T = +\infty$ is not excluded— and $k^*(t) = k(t - T)$ for $t > T$.

Then since on $[0, T]$

$$x(t) \leq h[k(t)] \leq h[k^*(t)] = x^*(t)$$

and

$$\lambda(t) = e^{-\int_0^t \rho(x(s))\, ds} \leq e^{-\int_0^t \rho(x^*(s))\, ds} = \lambda^*(t)$$

and

$$0 \leq u(x(t)) \leq u(x^*(t))$$

one has

$$\begin{aligned} W(0) - W^*(0) &= \int_0^T [\lambda(t)\, u(x(t)) - \lambda^*(t)\, u(x^*(t))]\, dt \\ &\leq 0 \end{aligned}$$

with equality only if $x^*(t) = x(t)$ for all t which is impossible since $h(k_0) < h(k_0^*)$. □

7.7 Optimal capital paths are strictly monotone (proof of proposition 6)

From what has been shown above, $\dot{k} = 0$ is possible for an optimal path only if it is stationary. □

7.8 Proof of proposition 7

To obtain a differential equation for x Differentiate the identity

$$u(x) - W\rho(x) + (u'(x) - W\rho'(x))(f(k) - x) = 0$$

This gives

$$((u'' - W\rho'')(f - x))\dot{x} + (-\rho - \rho'(f - x))\dot{W} + (u' - W\rho')f'\dot{k} = 0 \quad \text{or}$$
$$\dot{x} = -\frac{(u' - W\rho')}{(u'' - W\rho'')}(f' - \rho - \rho'(f - x)) \equiv \psi(k, x)h(k, x)$$

where

$$\psi(k, x) \equiv -\frac{(u' - \omega\rho')}{(u'' - \omega\rho'')} > 0$$
$$\omega(k, x) \equiv \frac{u(x) + u'(x)(f(k) - x)}{\rho(x) + \rho'(x)(f(k) - x)}$$

The paths for x, k can the be obained from the system

$$\begin{aligned} \dot{x} &= \psi(f' - \rho - \rho'(f - x)) \\ \dot{k} &= f - x \end{aligned}$$

with a stationary solution at any root of $g(k) \equiv f'(k) - \rho[f(k)] = 0$. At such a point, locally the linear approximation

$$\begin{pmatrix} \partial\dot{x} \\ \partial\dot{k} \end{pmatrix} = \begin{pmatrix} 0 & f'' - \rho' f' \\ -1 & f' \end{pmatrix} \begin{pmatrix} \partial x \\ \partial k \end{pmatrix}$$

has a stable saddle path only if the determinant of the system,

$$f''(k) - \rho'[f(k)]f'(k) = g'(k),$$

is negative. Thus if the stationary capital is approached from below, $k < k^S$ and increases, $\Rightarrow g(k) > g(k^S) = 0$. Thus $f'(k) > \rho[f(k)]$ □

7.9 Proof of proposition 8

This follows from the fact that optimal paths exist, and have to arrive at some stable saddlepath. Under the assumptions in the text, the function $g()$ must have, at least generically, one root at which it slopes downward. □

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Figure 1

Aggregator function F(x,W)

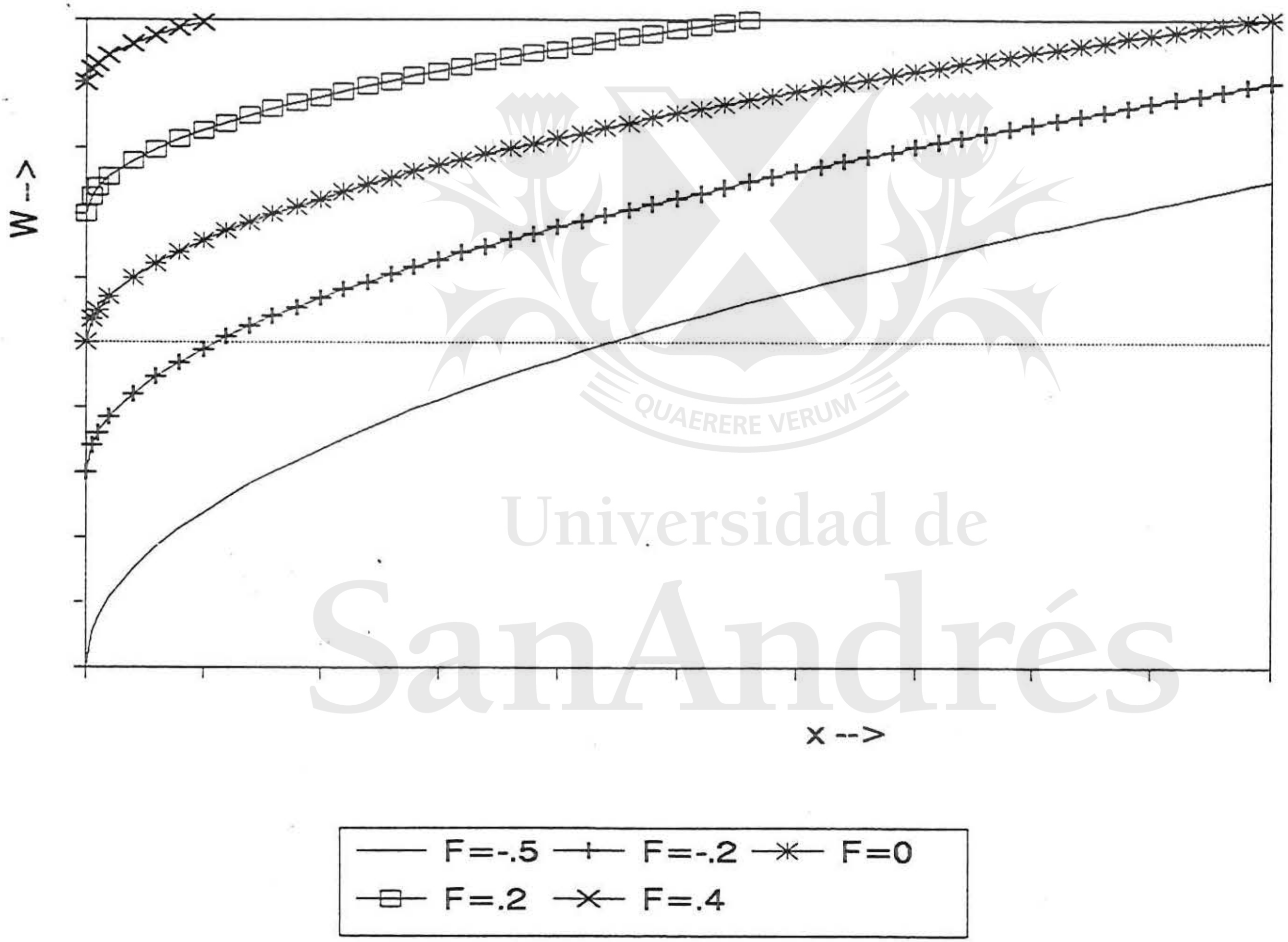
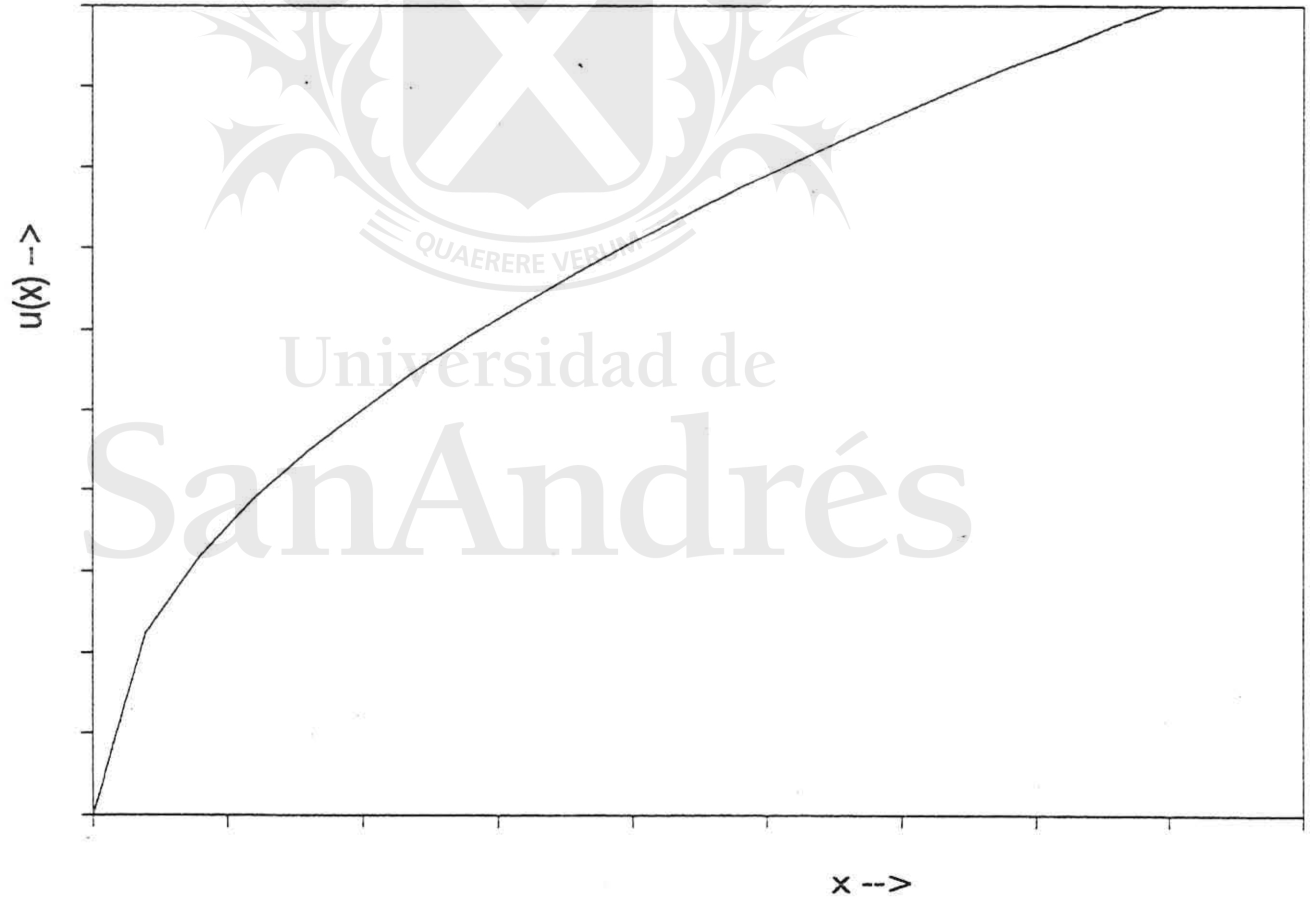


Figure 1b
Instantaneous utility function u(x)



2

Figure 1c

Rate of time preference function rho(x)

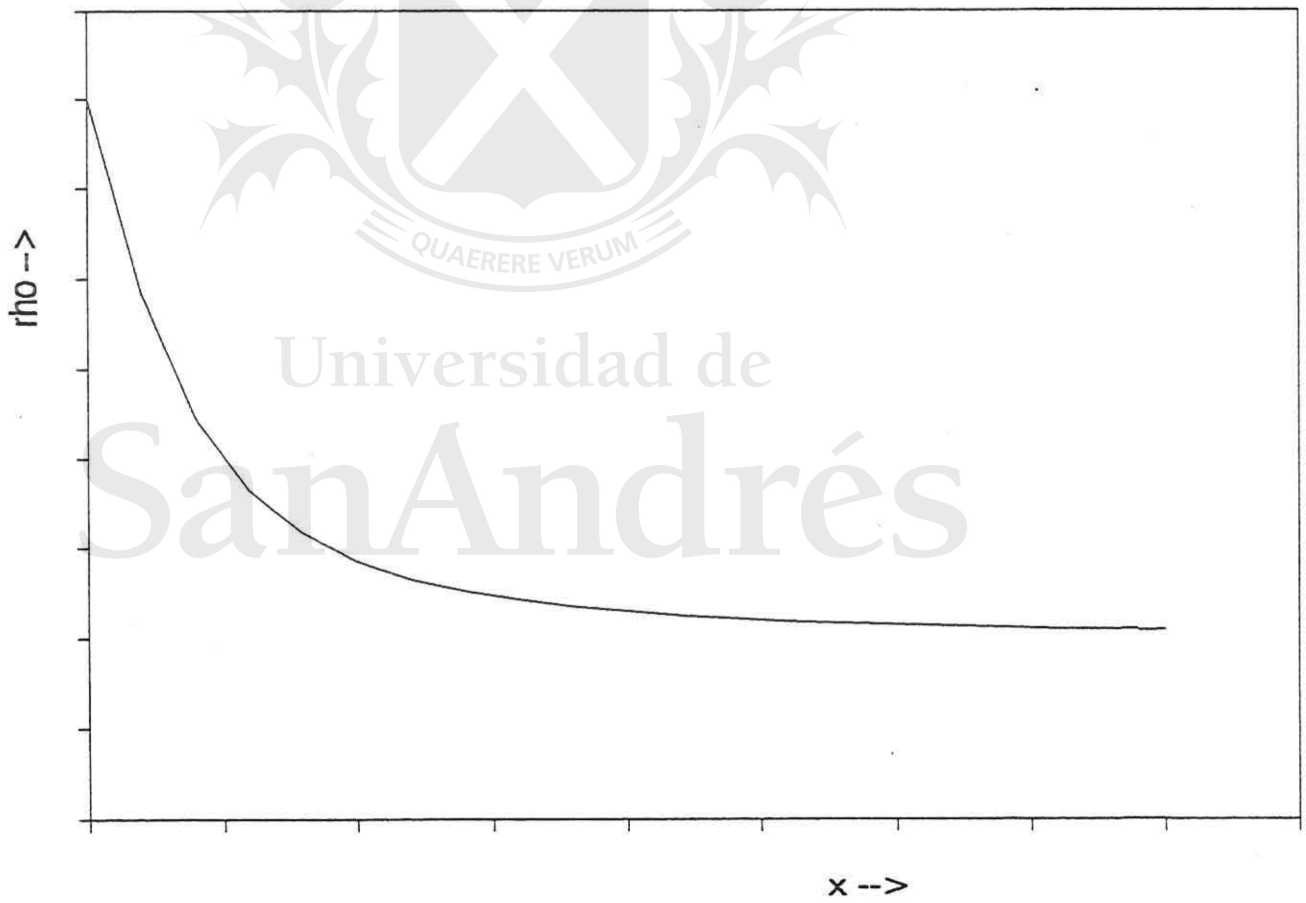


Figure 2

Production function f(k)

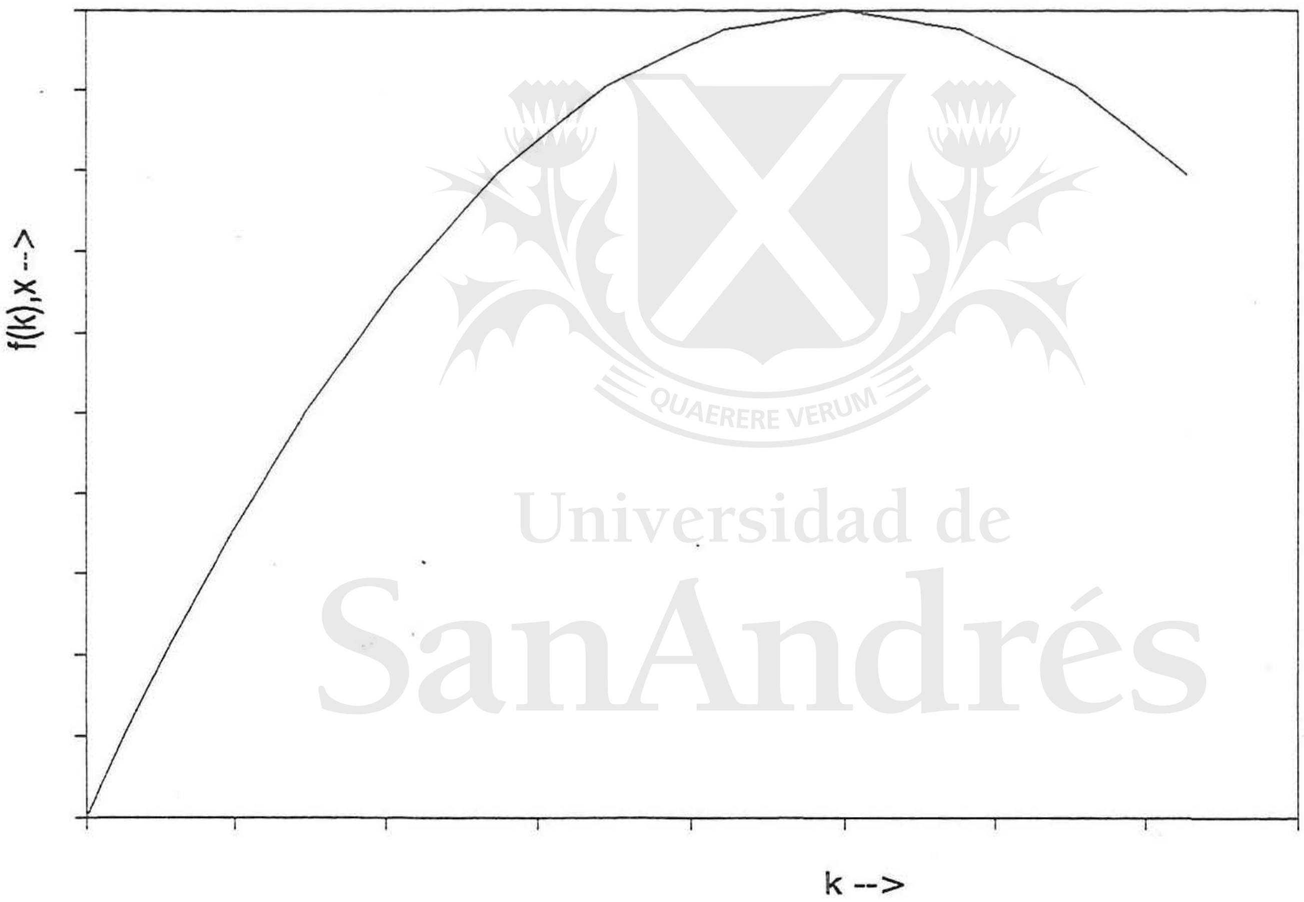


Figure 3
MPK and RTP

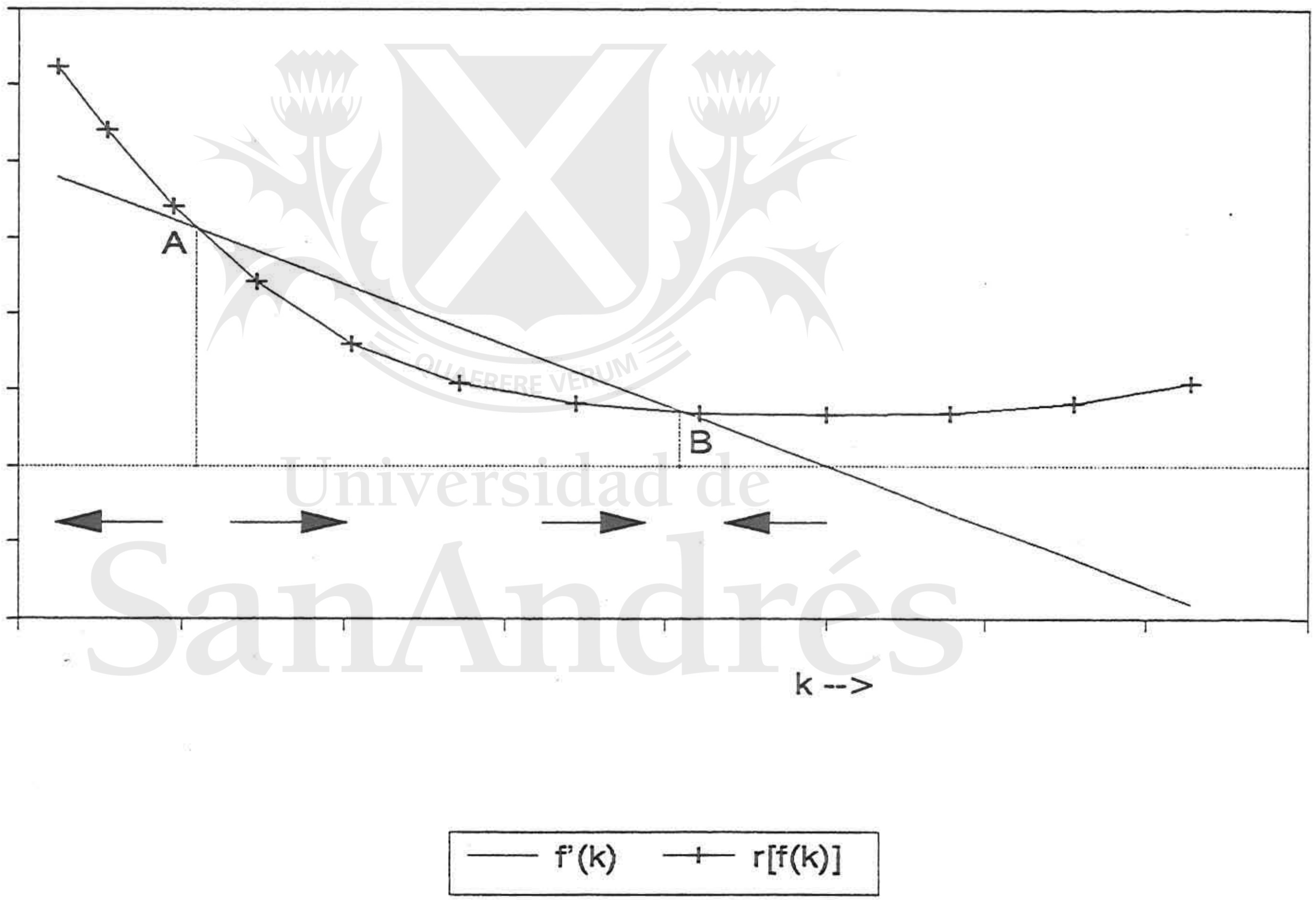
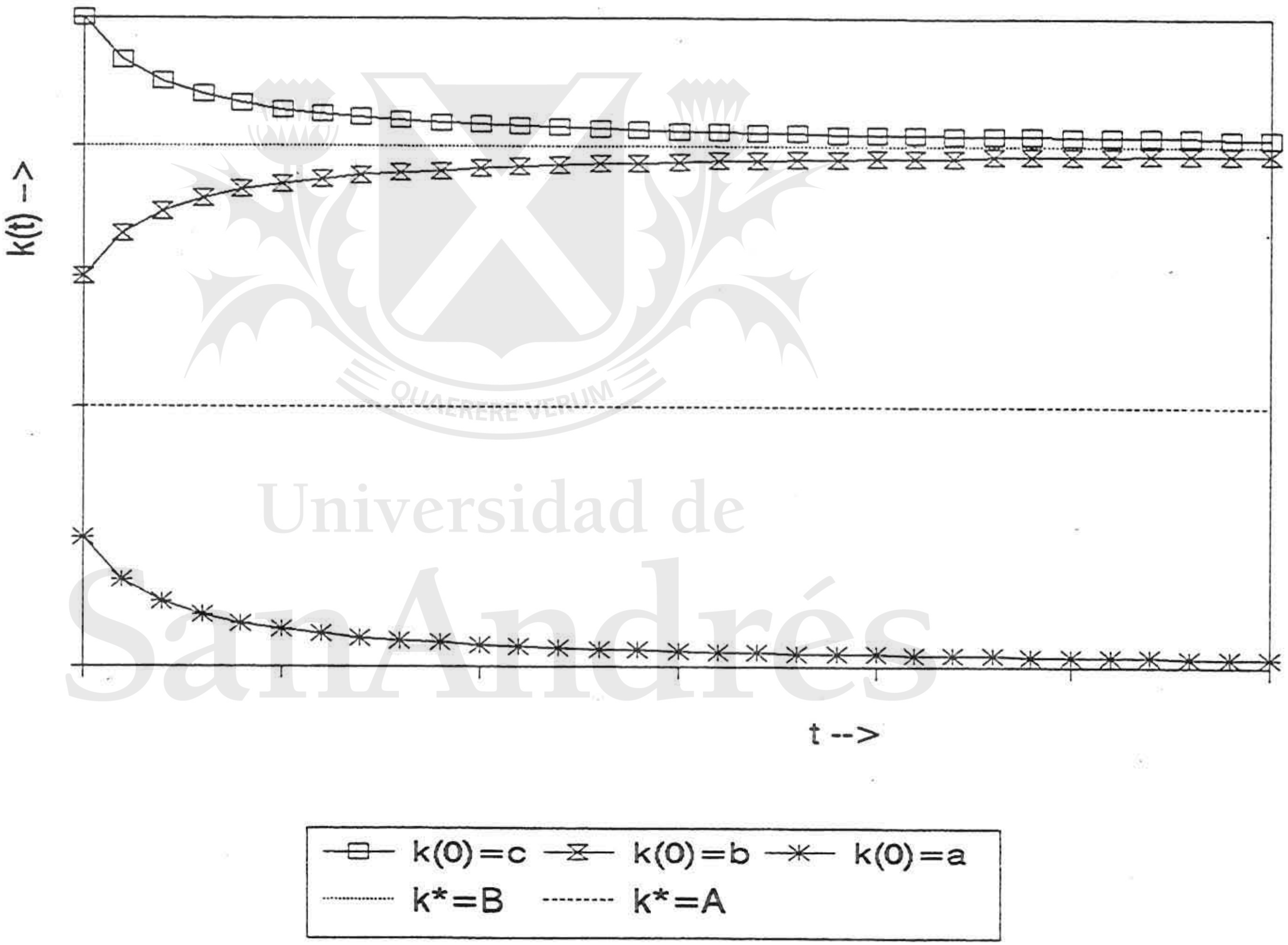
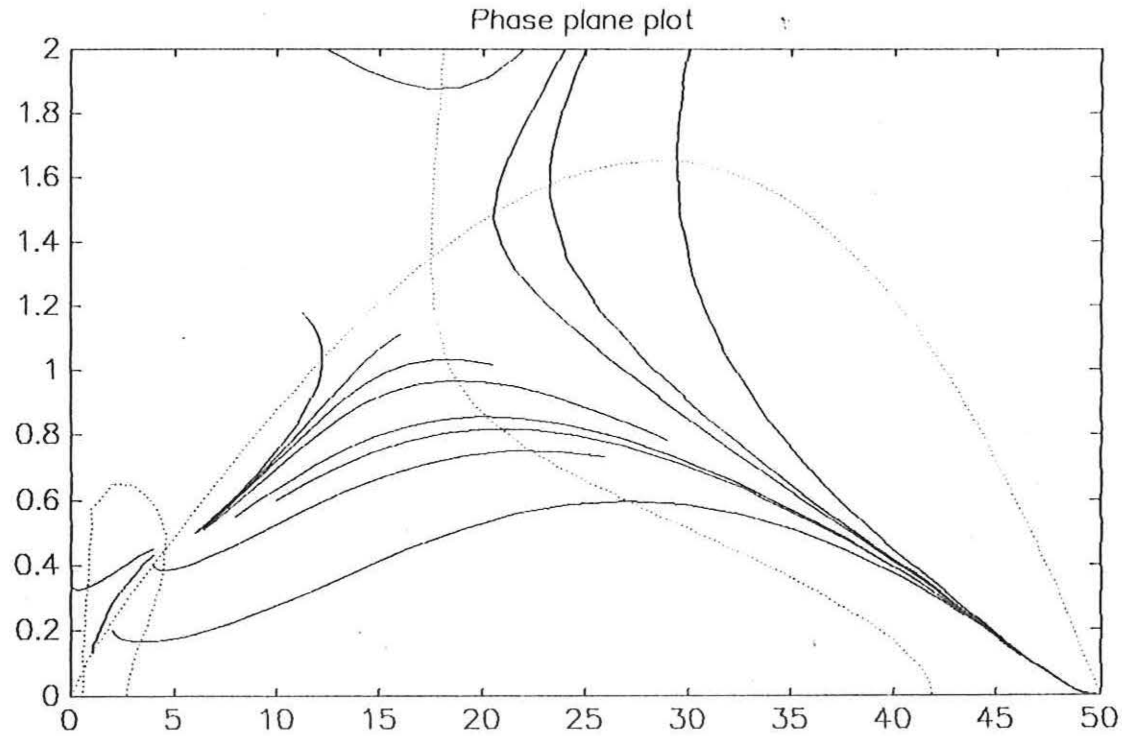


Figure 4
Capital programs k(t)




Phase plane plot
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1.8
1.6
1.4
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45
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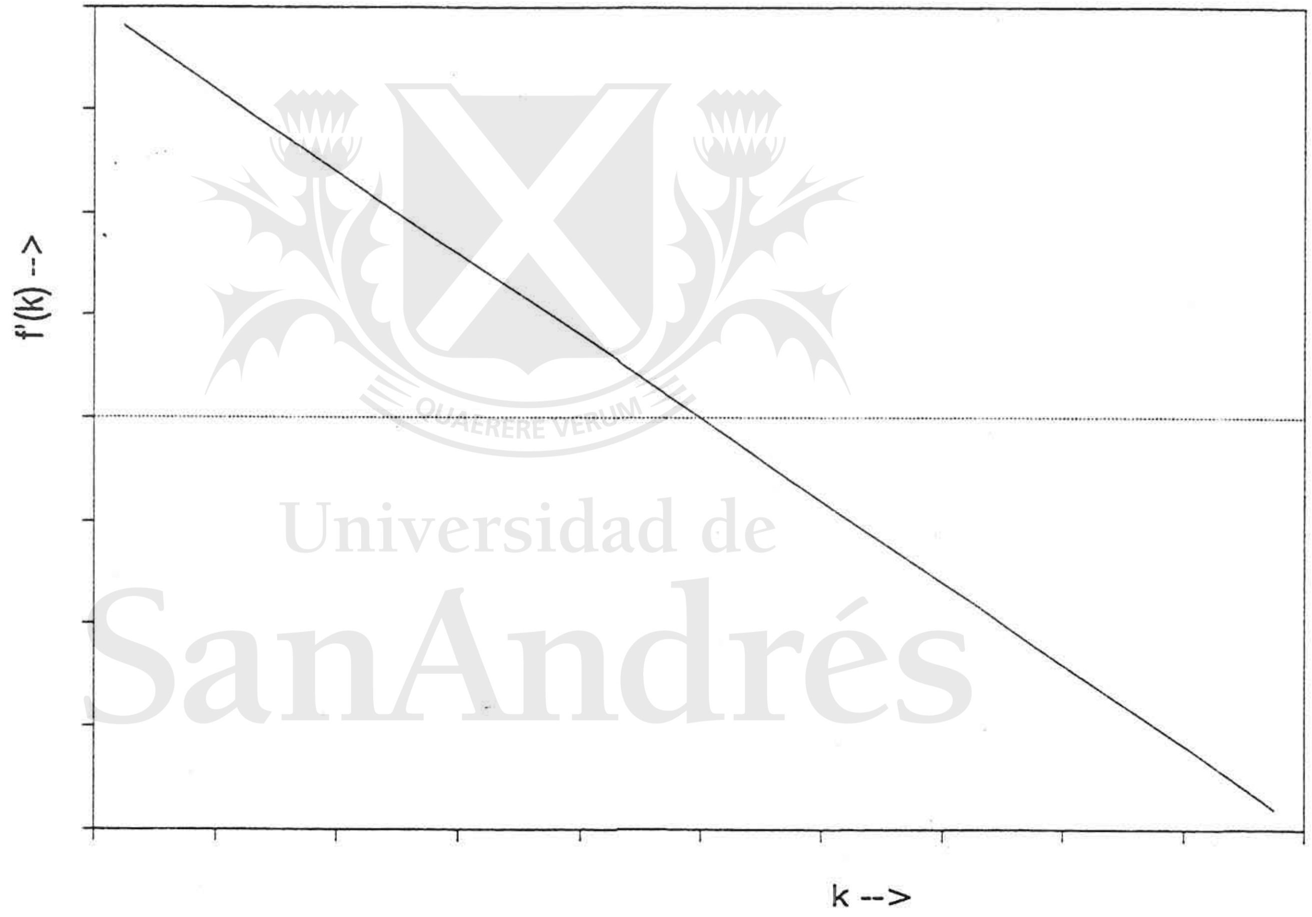
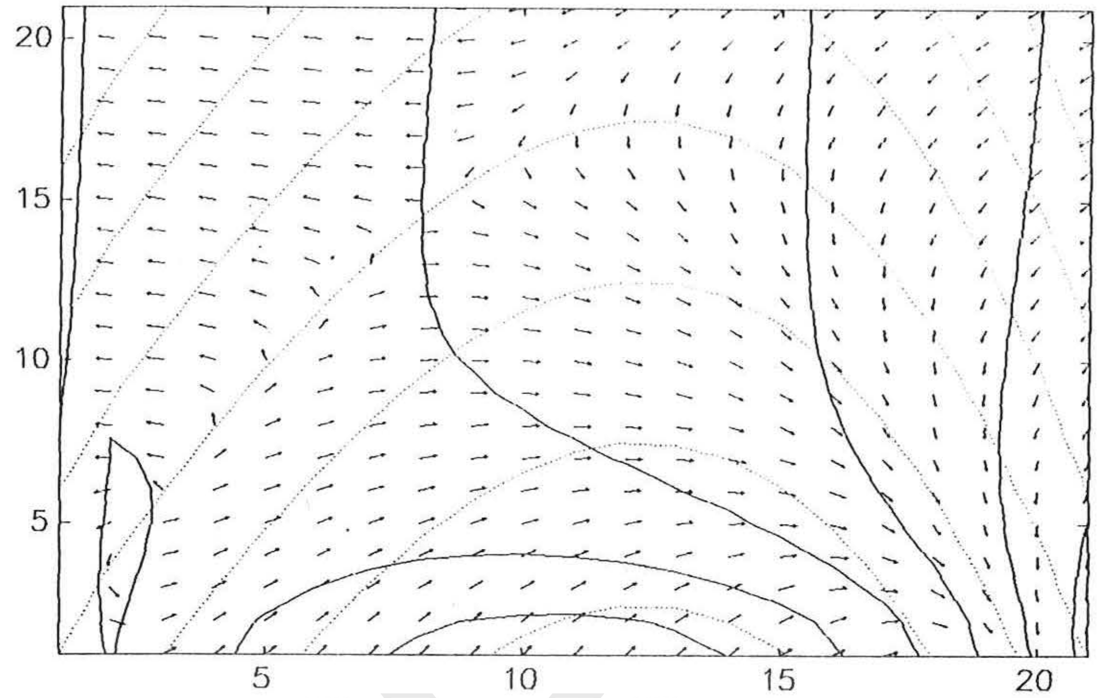
FIGURE 5

Marginal product of capital f'(k)





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