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Partial Preferential Orderings and Rationality

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Abstract

The notion of rational behavior is fundamental for disciplines like Economics, Psychology and Artificial Intelligence. It is usually represented by choice processes, involving connected orderings of alternatives, from which the maximal elements become chosen.

When this paradigm is applied to collections of agents its drawbacks manifest themselves, as shown by K.Arrow. As a result, the notion of rationality was considered as changing its meaning in the case of aggregates.

In this paper we will argue for a weakened notion of rationality, which shows a property of scale-invariance. The pros and cons of this approach will be examined in the light of its application to certain problems in Economic Theory and Artificial Intelligence.

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1 Introduction

The notion of Rationality underlies the representation of decision-making processes. In the case of an individual, who faces a set of alternatives, the rational action is to chose the "best" one. More precisely, the idea is to determine a binary relation P on the the set of alternatives, such that the choice consists of the subset of P-maximal alternatives [1]. The relation P is usually interpreted as a *preferential ordering*, in such a way that the choices are the *most prefered* options.

In Economics, the marginalist school introduced a functional representation of rational behavior. It required to take into account a real valued function on the set of alternatives, called an *utility* function. The best options obtained as the ones that maximized utility. Obviously, the existence of such a function was a strong requirement, which later was shown equivalent (up to monotone transformations), by means a nice representation theorem, to a preference relation with the properties of a *weak order*. That means (notwithstanding different nomenclatures in the literature), that the derived "as preferred" relation should be transitive and connected. So, a good deal of current Economic Theory has been constructed on the assumption that each economic agent has a weak ordering of preferences on the set of commodities [2]. The usefulness of this approach can be witnessed in the abundance of results that can be elegantly derived from the assumed features of rational agents.

The medieval problem of Buridan's ass, which starves in between two bales of hay, shows that, although appliable in many cases, the weak order approach to rationality can not handle certain others. In fact, this case is interesting as it shows the main drawback of using weak orders as representations of preferences: a pair of alternatives may not be comparable. In other words, connectedness is a very strong property that can not be easily assumed.

Given an aggregate of rational agents, it seemed natural that the notion of rational behavior should be the same as for individuals. The main problem was the determination of an aggregation process that provided any collective preferential ordering up from the preferential orderings of the agents. Despite the innocent-looking properties desired for this aggregation procedure, the well known *Arrow's Impossibility Theorem*, states that no collective ordering can be obtained satisfying the required conditions [3].

An elegant order-theoretic analysis of Arrow's conditions shows again the criticality of assuming weak orders as representations of preferences. It will be exhibed, in the following section, that a better option, is to assume either *partial* or *acyclic* preferential orderings. In the third section the consequences of relaxing the conditions on the preferences orderings will be examined. Then the consequences of these results will be considered for Economic Theory and Artificial Intelligence.

2 Aggregation of preferences

Scandinavian logicians have substantively contributed to the analysis of preferences and their aggregation. G.H.von Wright clarified the logical properties of preferences, emphasizing the derived property of *indifference* among alternatives, which in case of weak orders is an equivalence relation [4]. In a later contribution, this author discussed briefly Arrow's Theorem, noting that the notion of indifference defined by $xIy \equiv \neg(xPy) \wedge \neg(yPx)$ allows two interpretations: one impliying that both alternatives are equivalent and the other that both are *incomparable* [5]. In the case of weak orders both interpretations coincide but this is not necesarily so if other sorts of order relations are considered. von Wright conjectured that this point could be significative for the problem of aggregation of preferences.

Actually, von Wright's conjecture was proved true by B.Hansson, who showed the rich order-theoretic structure that underlies Arrow's proof. To consider in detail Hansson's insights lets state precisely Arrow's aggregation problem [3]:

Definition 1 *Given a set* $\{\prec_i\}$ *of binary order relations on a set of* alternatives S, *for* $i \in I$, *a set of individuals or criteria, it has to be defined a* collective order relation $\prec_I$ *using an* aggregative mapping σ *such that* $\sigma(\{\prec_i\}) = \prec_I$, *verifying the following conditions:*

1. **Positive responsiveness** $\forall a, b \in S$ *and* $\{\prec_i\}, \{\prec_i^*\}$ *such that* $\{i \in I : a \prec_i b\} \subseteq \{i \in I : a \prec_i^* b\}$, $a\sigma(\{\prec_i\})b \Rightarrow a\sigma(\{\prec_i^*\})b$

2. **Independence of irrelevant alternatives** $\forall a, b \in S$, $\{\prec_i\}, \{\prec_i^*\}$ *such that* $\{\prec_i\} \equiv \{\prec_i^*\}$ *on* $\{a, b\}$, $\sigma(\{\prec_i\}) \equiv \sigma(\{\prec_i^*\})$ *on* $\{a, b\}$

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3. **Pareto principle** $\forall a, b \in S$ *if* $\forall i \in I\ a \prec_i b$ *then* $a \prec_I b$

4. **Nondictatorship** $\nexists i_0 \in I$ *such that* $\forall a, b \in S,\ a \prec_{i_0} b \Rightarrow a\sigma(\{\prec_i\})b$

To make more precise the order relations above lets give the following characterizations [6]:

Definition 2

- $\prec$ *is a* partial order *if*
  - $\forall a \in S,\ a \not\prec a$ *(irreflexivity)*
  - $\forall a, b, c \in S,\ a \prec b \wedge b \prec c \Rightarrow a \prec c$ *(transitivity)*
- $\prec$ *is a* weak order *if*
  - $\forall a, b \in S,\ a \prec b \Rightarrow b \not\prec a$ *(asymmetry)*
  - $\forall a, b, c \in S,\ a \prec b \Rightarrow (a \prec c \vee c \prec b)$ *(negative transitivity)*
- $\prec$ *is an* acyclic order *if*
  - $\forall_1 \ldots a_n \in S\ a_1 \prec a_2 \wedge a_2 \prec a_3 \wedge \ldots \wedge a_{n-1} \prec a_n \Rightarrow a_n \not\prec a_1$

Lets note that from the *indifference* relations, defined $aIb \equiv a \not\prec b \wedge b \not\prec a$, being only an equivalence relation the corresponding to a partial order $\prec$. As was said above, K.Arrow proved the following:

Theorem 1 *There exists no σ satisfying the conditions* 1, …, 4 *above if* $|S| \geq 3$ *and the* $\prec_i$ *are* weak orders

In the proof of this theorem plays a significative rôle the notion of *decisive set*:

Definition 3 $G \subseteq I$ *is decisive iff* $\forall a, b \in S,\ a \prec_{i \in G} b \Rightarrow a \prec_I b$

The class of decisive sets may show interesting properties:

Definition 4 $\Omega = \{G \subseteq I : G \text{ is decisive}\} \subseteq 2^I$, *has a natural order by inclussion. It is called a* prefilter *iff*

- $I \in \Omega$
- *if* $A \in \Omega$ *and* $A \subseteq B$, *then* $B \in \Omega$
- *if* Γ *is a finite subfamily of* Ω, $\cap\Gamma \neq \emptyset$

it is called a filter *if it additionally verifies*

- $\forall A, B \in \Omega\ ,\ \ A \cap B \in \Omega$

and an ultrafilter *if it also verifies*

- $\forall A \subseteq I, A \in \Omega \vee \bar{A} \in \Omega$.

In fact, Hansson's approach to the aggregation problem is through the properties of Ω. The conditions $1, \ldots, 3$, when the individual orders are weak, imply that Ω is an *ultrafilter*. A general property of ultrafilters states that in case that I is finite, exists a $i_0 \epsilon I$ such that $\cap\Omega = i_0$. Of course, this i_0 is the undesired dictator.

A way out of the conclusions of the Impossibility Theorem is to consider the other kinds of order relations. Hansson showed that if the individual orders are *partial* Ω is a *filter*, so no dictator appears and a collective partial order obtains. D.Brown, following this analysis proved that, if Ω is a *prefilter* it aggregates *acyclic* orders in an *acyclic* collective order, but the converse is only true when S is infinite [3].

3 A scale invariant notion of rationality

In the physical sciences a way to look at complex phenomena is to consider them as the result of an aggregation of elemental phenomena. It is obvious that in many cases this allows to explain the emergence of properties in an aggregation level, up from an underlying level which lacks these properties. But another aim in a theoretical inquiry is to detect *universal properties*, which are *scale invariant*. It is not easy to determine the existence of such properties, given that it involves to prove that in the framework of a theory founded in the low-level phenomena there are properties preserved by the aggregation process. This difficulty is largely rewarded in case of success, given that it shows that the theory obeys to the fundamental epistemological principle of *parsimony*, also known as *Okham's Razor* [7].

It was noted in the last section that, for the aggregation of preferences, only in the case of partiality exists a scale invariance. The relaxed notion of rationality represented by partial ordered preferences is preserved when collective rationality is considered. As said above, this is an epistemologically elegant solution to the aggregation problem. Moreover, to prove that this was the case (for the weak order notion of rationality) was the original aim of O.Helmer when he posed the problem to K.Arrow [8].

To make more precise the notion of scale invariance lets give it a category theoretic flavor [9]:

Definition 5 *A* category C *is a set of* objects $P_1, P_2, \ldots$ *and* morphisms $f_1, f_2, \ldots$ *among them, including, for each object its* identity *morphism. There is also defined an associative* composition *of morphisms* $\circ$.

An example of category is $Posets$, whose objects are partial orderings of a set S. The corresponding morphisms are the order preserving mappings, i.e. if P_1 and P_2 are two objects of the category, $f : P_1 \rightarrow P_2$ a morphism it verifies $a \prec_{P_1} b \ \Rightarrow \ f(a) \prec_{P_2} f(b)$. Another example of category is the *cartesian product* of a subset indexed by I of elements of $Posets$, $\prod_I P_i$, and the corresponding morphisms are naturally the products of the order preserving morphism of among the P_i. This product can be called, in the language of Social Choice Theory a *profile*.

Definition 6 *Given two categories* C_1 *and* C_2*, a homomorphism*
$\rho : \ C_1 \rightarrow C_2$
preserving the composition of morphisms is called a functor

Several authors emphasized that the existence of a functor between a cartesian category and the original category implies an universal property. The argument is that cartesian categories represent an aggregate, and such a functor preserves the morphisms, which become an invariant at changes of scale [9]. So, the notion of rationality as a scale invariant property in the case of partial orders can be infered from the following theorem:

Theorem 2 *The decisive sets class* Ω *forms a filter iff there exists a functor* $\rho : \ \prod_I \ P_i \rightarrow Posets$

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4 Applications

The source of the failure of the usual representation of rationality is the assumption that any rational agent has a connected ordering of preferences. Instead, if the preferences are partialy ordered, the derived indifference relation is not transitive, showing that von Wright's conjecture was correct, in the sense that in this case no impossibility result follows.

A main drawback of representing preferential orderings is the difficulty of defininig an utility function as usual in Economic Theory. B. Peleg studied this problem and proved a representation theorem for partial orderings[10]. Before stating this result, certain notions should be defined:

Definition 7 *A partial order $\prec$ defined on a topological space S is*

- continuous *if* $\forall a \in S\ L(a) = \{x : x \prec a\}$ *is an open set in the topology*
- separable *if* $\forall a, b \in S,\ a \prec b \Rightarrow \exists c$ *such that* $a \prec c \prec b$
- spacious *if* $a \prec b \Rightarrow L^c(a) \subset L(b)$

A generalization of Peleg's theorem is the following

Theorem 3 *If*

- *S is a countable set or*
- *if it is an uncountable topological space with $\prec$ a continuous, separable and spacious partial order*

there exists a real valued function U on S such that $a \prec b \Rightarrow U(a) < U(b)$ and in the second case it is also continuous

An utility function as the implied by Peleg's result lacks the nice properties of the usual utility function of agents in an Arrow-Debreu market. Considering a fundamental axiom of consumer behavior, *convexity* of preferences which implies that $\forall a \in S$, $G(a) = \{x \in S : a \prec x\}$ is convex, it is clear that this property is in general not valid for partial orders. Given two $b, c \in G(a)$ nothing implies that $[b, c] \in G(a)$. Related with this result is also the fact that a maximization of U provides in general a set of "indifferent" elements, meaning in this case the incomparability of options. An extra element of choice has to be added to U if only one option should be chosen.

It is not necessary here to go further in the points of conflict between the usual assumptions of consumer behavior and the properties derived from a partial ordering of preferences. This is not, despite the solidity of the Arrow-Debreu framework, a valid argument for discarding partial orderings, but for deepening this inquiry and develop an alternative framework.

Another area of application of this representation of rationality is *Artificial Intelligence*. In this field, a main goal is to develop a computational "reasoner", which confronted with the inconsistencies of *common sense* knowledge should give correct answers to the queries posed to it. To do so involves to chose among several alternatives, each one incompatible with the others. The usual formal framework of reasoning is given by *deductive systems*, consisting in a set of *axioms* and *rules of inference* formulated in a formal language. Applying iteratively the rules on the axioms, obtains *theorms* which are as valid as the axioms. In case of inconsistency, every sentence of the formal language become theorems, meaning that the system is *trivial*.

The problem with deductive systems as representations of common sense reasoning is that no useful formal language can avoid the apparition of inconsistences. A recurrent example consists in the following set of sentences: *"birds fly"*, *"penguins are birds"*, *"penguins don't fly"*. To avoid the trivialization of a reasoner, the notion of deductive systems has been extended to that of *information systems*, whose main differences with the formers is the existence of several *maximally consistent* sets of theorems or *extensions*.

In the framework of information system, several methods have been developed for answering a query. For each one, the correct answer pertains simply to the most preferred extension. It is obvious that the way the extensions are preferentially ordered may change the answer. Given that this methods represent different criteria, the ultimate reasoner seeked by Artificial Intelligence should unify these criteria.

J.Doyle analyzing the possibility of an unified reasoning framework, stated an analogous of Arrow's theorem where the criteria are given by the different reasoning methods. To clarify the matter, it must be considered that each method provides, implicitly, a partial preferential ordering of extensions [11]. Then, the universal reasoner should provide a global ordering of extensions, such that the answers to queries are given by the most preferred extensions.

Doyle argued for an aggregative process to obtain the global ordering.

This process should obey the conditions $1,\ldots,4$ above, which provide "equal opportunity" to each criteria. It should be noted that Doyle, although aknowledging that the individual orders are partial, requires that the global one should be a weak order. This is, we think, an unnecesary requirement as shown by the also recurrent *Nixon diamond* example. Given the statements *"Quakers are pacifists"*, *"Republicans are not pacifists"*, *"Nixon is quaker"*, *"Nixon is republican"*, no definite answer can be given to the query *"Is Nixon pacifist?"*. It may argued that the set of statements is not informative enough, but this is not a problem of the reasoner for which this set is an input.

Examples like Nixon's diamond show that a positive answer may not be preferable to the negative one and viceversa. A partial global order seems so more appropiate than a weak one. This calls again for a notion of rationality of reasoning represented by partial orders. Doyle instead states his aggregation problem and formulates an impossibility theorem implying that partial orders can not be aggregated in a weak order. This result is false, given that it means that deriving a dictatorial result as Doyle does implies that Ω is an *ultrafilter*. But as was said Ω forms a *filter* in the case of individual partial orders. So Doyle's Impossibility Theorem means that there are no *proper filters* i.e. that each filter is an ultrafilter [12].

Doyle's error should not be taken as meaning that partial orders can be in every case be aggregated in a weak order. The solution lies in assuming also a partial global order of extensions. So the possibility of an universal reasoner remains open.

5 Conclusions

It was shown that rationality can be considered as represented by partial preferential orderings. This conception helps to show how to solve several problems generated by the more restrictive representation by weak orders. But it opens also new problem, mainly for Economic Theory. In this field, several axioms of the consumer behavior are not more tenable, calling for a monumental task *à la* Debreu, in a much harder terrain.

Nothing was said about choice under *uncertainity*. But formally this problem can be handled assuming that S is a set of *probability distributions* on an underlying set of alternatives. All the arguments of former sections

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are appliable in this case, showing again the pertinence of an observation of von Wright. This author noted that the notion of indifference among lotteries can have two meanings as preferences under certainity [5].

Finally, it is interesting to consider the possibility of incorporate in the notion of choice, exogenous factors on preferences. Two prospects to be explored are:

- Random choice among maximally preferred alternatives
- Metarationality: choice among contrafactual choices

The last one means simply that given a set of possible global choices, an agent examines them and choses the best option in order to help to obtain the most desired global ones. This approach can be directly related with Game Theory and may be interesting to use the famous *Prisoner's Dilemma* as a benchmark [13].

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