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von Neumann algebra”**

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# Metric geometry of partial isometries in a finite von Neumann algebra\*

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## Abstract

We study the geometry of the set

$$\mathcal{I}_p = \{v \in M : v^*v = p\}$$

of partial isometries of a finite von Neumann algebra  $M$ , with initial space  $p$  ( $p$  is a projection of the algebra). This set is a  $C^\infty$  submanifold of  $M$  in the norm topology of  $M$ . However, we study it in the strong operator topology, in which it has not a smooth structure. This topology allows for the introduction of inner products on the tangent spaces by means of a fixed trace  $\tau$  in  $M$ . The quadratic norms define not a Hilbert-Riemann metric, for they are not complete. Nevertheless certain facts can be established: a restricted result on minimality of geodesics of the Levi-Civita connection, and uniqueness of these as the only possible minimal curves. We prove also that  $(\mathcal{I}_p, d_g)$  is a complete metric space, where  $d_g$  is the geodesic distance of the manifold (or the metric given by the infima of lengths of piecewise smooth curves).

**Keywords:** partial isometries, projections, geodesics, finite von Neumann algebras.

## 1 Introduction

Let  $M$  be a finite von Neumann algebra with fixed trace  $\tau$ , and let  $p \in M$  be a projection. Consider the set

$$\mathcal{I}_p = \{v \in M : v^*v = p\}$$

of partial isometries in  $M$  with initial space  $p$ . This set is a  $C^\infty$  differentiable submanifold of  $M$  in the norm topology, and a homogeneous space of the unitary group  $U_M$  of  $M$ , via the action

$$u \cdot v = uv, \quad u \in U_M, \quad v \in \mathcal{I}_p.$$

This action is locally transitive: if  $v, v_0 \in \mathcal{I}_p$  with  $\|v - v_0\| < 1/2$ , then there exists a unitary valued  $C^\infty$  map  $\omega = \omega(v, v_0)$  such that  $\omega v_0 = v$  (see [10], [3] and [2] for an account of these facts). In the case of a general von Neumann algebra the action is not transitive. For instance, in the space of isometries (initial space equal to 1) in the algebra  $\mathcal{B}(H)$  of all bounded operators in a (infinite dimensional) Hilbert space  $H$ , the action preserves the Fredholm index, so it cannot be transitive. However, when  $M$  is finite, it is transitive. Indeed, considering the right  $pMp$ -Hilbert  $C^*$ -module  $pM$  with the inner product  $\langle x, y \rangle = x^*y$ , the space  $\mathcal{I}_p$  is what in [4] was called the unit sphere

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of the module (i.e. the set of elements  $x$  such that  $\langle x, x \rangle = p$ , the unit of  $pMp$ ). In that paper it was proved that the unit sphere is connected when the algebra is finite, and therefore the action is transitive.

Although  $\mathcal{I}_p$  has a smooth structure only in the norm topology, we shall be mainly concerned with the strong operator topology in  $\mathcal{I}_p$ . We introduce a metric in the tangent spaces of  $\mathcal{I}_p$ , the inner products defined are computed by means of the action of  $U_M$ , and in terms of the trace  $\tau$  of  $M$ . The quadratic norms in the tangent spaces (which are real linear subspaces of  $M$ ) are equivalent to the 2-norm of  $\tau$ . Therefore the Riemannian metric is not complete, the tangent spaces are only pre-Hilbert spaces, and this setting differs from the classical Riemann-Hilbert theory of manifolds. In fact the set  $\mathcal{I}_p$  is not even a manifold in the strong operator topology. Nevertheless we proceed with the geometric study of  $\mathcal{I}_p$ , with ad-hoc as well as classical tools. Our main interest is the metric space  $(\mathcal{I}_p, d_g)$ , where  $d_g$  is the metric given by the infima of the lengths of curves joining two given points in  $\mathcal{I}_p$ , measured with the Riemannian metric.

In section 2 we introduce the metric and compute its Levi-Civita connection. This connection was previously studied in the general setting of homogenous reductive space [11], where it was labeled the *classifying connection* of the homogeneous space. There the geodesics of this connection were computed. Also in section 2 we transcribe previous results [1], [5], [6] on the geometric structure of the unitary group in the strong operator topology. Our results on minimality of geodesics of the connection rely on these results, particularly on the convexity properties obtained in [6]. We prove that there exists a radius  $R > 0$  such that for any  $v \in \mathcal{I}_p$ , the ball (in the usual norm  $\| \cdot \|$  of  $M$ )  $\mathcal{E}_v(R)$  centered at  $v$  has the following property: for any  $v' \in \mathcal{E}_v(R)$  there exists a geodesic joining  $v$  and  $v'$ , which is shorter than any other piecewise smooth curve  $\nu(t)$  *inside*  $\mathcal{E}_v(R)$  joining the same endpoints. Note that this result falls short from saying that  $\mathcal{E}_v(R)$  is a normal neighbourhood of  $v$  in two aspects: first,  $\mathcal{E}_v(R)$  is not open in the strong operator topology, second, the geodesic is minimal only among curves inside  $\mathcal{E}_v(R)$ . Could there be a shorter curve wandering outside  $\mathcal{E}_v(R)$ ? We do not know the answer to this question. Since  $\mathcal{I}_p$  is far from being a Riemann-Hilbert manifold, weaker results than in the classical setting are to be expected. Concerning minimality, we show that if a piecewise curve is minimal, then it must be a geodesic of the linear connection. Another question of which we do not know the answer: does there always exist a minimal curve between two points? In section 3 we prove that  $(\mathcal{I}_p, d_g)$  is a complete metric space.

In  $\mathcal{I}_p$ , the strong operator topology is metrizable by the 2-norm given by the trace. This fact is certainly known, let us finish this introduction by giving a proof of it. Recall that

$$\|x\|_2 = \tau(x^*x)^{1/2}.$$

The usual norm of  $M$  will be denoted by  $\| \cdot \|$ .

**Lemma 1.1** *The 2-norm  $\| \cdot \|_2$  metrizes the strong operator topology in  $\mathcal{I}_p$ . The metric space  $(\mathcal{I}_p, \| \cdot \|_2)$  is complete.*

**Proof.** Let  $H = L^2(M, \tau)$  be the Hilbert space obtained by completion of  $(M, \| \cdot \|_2)$ , and suppose  $M$  represented in  $H$  (by left multiplication). The elements  $x \in M$  when regarded as vectors in  $H$  will be denoted  $\xi_x$ , and as operators in  $H$  they will be denoted by  $l_x$ . Note that the commutant of  $M$  in this representation consists of right multiplication operators  $r_x$ , for  $x \in M$ :  $r_x$  is the completion of  $r_x \xi_y = \xi_{yx}$ . Suppose that  $v_n \rightarrow v$  strongly in  $\mathcal{I}_p$ . Note that

$$\|v_n - v\|_2^2 = \tau((v_n - v)^*(v_n - v)) = 2\tau(p) - \tau(v_n^*v) - \tau(v^*v_n).$$

Then  $\tau(v_n^*v) = \langle \xi_v, l_{v_n} \xi_1 \rangle \rightarrow \langle \xi_v, l_v \xi_1 \rangle = \tau(v^*v) = \tau(p)$  and  $\tau(v^*v_n) = \overline{\tau(v_n^*v)} \rightarrow \tau(p)$ , so that  $\|v_n - v\|_2 \rightarrow 0$ . Conversely, suppose that  $\|v_n - v\|_2 \rightarrow 0$ . Clearly this can be read as  $\|l_{v_n} \xi_1 - l_v \xi_1\|_H \rightarrow 0$ .

$$l_{v_n} r_x \xi_1 = r_x l_{v_n} \xi_1 \rightarrow r_x l_v \xi_1 = l_v r_x \xi_1.$$

The linear subspace  $\{r_x \xi_1 : x \in M\}$  is dense in  $H$  ( $\xi_1$  is cyclic and separating for  $M$ ), therefore  $l_{v_n} \xi \rightarrow l_v \xi$  for all  $\xi \in H$ , because the sequence  $v_n$  is bounded in norm. Let us show now that  $(\mathcal{I}_p, \|\cdot\|_2)$  is complete. Let  $v_n$  be a Cauchy sequence in  $\mathcal{I}_p$  for the 2-norm. Then there exists  $\xi \in H$  such that  $\xi_{v_n} \rightarrow \xi$ . every element  $\xi \in H$  defines a possibly unbounded operator  $l_\xi$  on  $H$ , whose domain includes  $\{\xi_x : x \in M\}$ :  $l_\xi \xi_x = r_x \xi$ . Then for all  $x \in M$ ,

$$l_{v_n} \xi_x = r_x \xi_{v_n} \rightarrow r_x \xi_v = l_\xi \xi_x. \quad (1.1)$$

Since  $v_n$  are partial isometries with initial space  $p$ , this implies that

$$\|l_p \xi_x\| = \|l_{v_n} \xi_{px}\| \rightarrow \|l_\xi \xi_{px}\| = \|l_\xi l_p \xi_x\|.$$

It follows that  $l_\xi$  acts isometrically in  $\{\xi_{px} : x \in M\}$  which is a dense linear subspace in the range of  $l_p$ . Analogously it can be proved that it acts trivially in  $\{\xi_{(1-p)x} : x \in M\}$ , which is dense in the range of  $I - l_p$ . Thus  $l_\xi = l_v$  is a partial isometry with initial space  $p$ , and (1.1) above implies that  $v_n \rightarrow v$  strongly.  $\square$

## 2 The metric induced by the reductive structure

Denote by  $V_v$  the isotropy group of the action, i.e. the subgroup of unitaries which fix  $v$ :

$$V_v = \{w \in U_M : wv = v\},$$

and denote by  $\mathcal{V}_v$  its Banach-Lie algebra, which consists of

$$\mathcal{V}_v = \{x \in M_{ah} : xv = 0\}.$$

Here  $M_{ah}$  denotes the real Banach space of antihermitic elements of  $M$ , which identifies with the Lie algebra of  $U_M$ . Denote by

$$L_v : U_M \rightarrow \mathcal{I}_p, \quad L_v(u) = uv$$

which is a submersion, and let

$$\ell_v = d(L_v)_1 : M_{ah} \rightarrow (T\mathcal{I}_p)_v, \quad \ell_v(x) = xv.$$

Note that the range of  $\ell_v$  is  $(T\mathcal{I}_p)_v = \{xv : x^* = -x\}$  and its kernel is  $\mathcal{V}_v$ .

There is a natural reductive structure for this homogeneous space ([8] [13]): a smooth distribution of horizontal spaces  $\{\mathcal{H}_v : v \in \mathcal{I}_p\}$  which are supplements for  $\mathcal{V}_v$ ,  $v \in \mathcal{I}_p$ :

$$\mathcal{H}_v \oplus \mathcal{V}_v = M_{ah},$$

which are invariant under the inner action of  $V_v$ :

$$w\mathcal{H}_v w^* = \mathcal{H}_v, \quad w \in V_v.$$

Let us choose these supplements  $\mathcal{H}_v$ . Denote by  $p^v = vv^*$  the final projection of  $v$ . An antihermitic element  $x \in \mathcal{V}_v$  satisfies  $xv = 0$  or equivalently  $xp^v = 0$ , and therefore if one represents it as a  $2 \times 2$  matrix (in terms of  $p^v$ ), it is of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & x_0 \end{pmatrix}$$

with  $x_0$  anti-hermitic. A natural supplement for  $\mathcal{V}_v$ , which is apparently  $\tau$ -orthogonal to  $\mathcal{V}_v$ , is the space  $\mathcal{H}_v$  which consists of antihermitic elements of  $M$  whose matrices, in terms of  $p^v$ , are of the form

$$\begin{pmatrix} z_{11} & z_{12} \\ -z_{12}^* & 0 \end{pmatrix}.$$

That is

$$\mathcal{H}_v = \{z \in M : z^* = -z, (1 - p^v)z(1 - p^v) = 0\}.$$

We introduce an incomplete Riemannian metric in  $\mathcal{I}_p$  by means of the trace  $\tau$ , and compute its Levi-Civita connection. The relevant data which encode all the information of the reductive structure are the coordinate maps

$$\kappa_v : (T\mathcal{I}_p)_v \rightarrow \mathcal{H}_v,$$

which are the inverses of the isomorphisms

$$\ell_v|_{\mathcal{H}_v} : \mathcal{H}_v \rightarrow (T\mathcal{I}_p)_v.$$

Explicitly:

$$\kappa_v(x) = xv^* - vx^* - p^v xv^*.$$

Consider in  $(T\mathcal{I}_p)_v$  the inner product which makes the map  $\kappa_v$  an isometry, when  $\mathcal{H}_v \subset M_{ah}$  is considered with the trace inner product:

$$\langle x, y \rangle_v = \tau(\kappa_v(y)^* \kappa_v(x)), \quad x, y \in (T\mathcal{I}_p)_v. \quad (2.2)$$

Therefore

$$\|x\|_v = \|\kappa_v(x)\|_2.$$

These norms can be computed explicitly, and they equal (after routine calculations, which involve the identities  $xp = x$  and  $x^*v + v^*x = 0$  valid for  $x \in (T\mathcal{I}_p)_v$ ):

$$\|x\|_v^2 = 2\tau(x^*x) - \tau(p^v xx^* p^v). \quad (2.3)$$

Note that this metric in  $(T\mathcal{I}_p)_v$  is equivalent to the trace inner product metric, though we claim that it is geometrically more relevant to the homogeneous structure. Indeed

$$\|x\|_v^2 = \tau(x^*x) + (\tau(xx^*) - \tau(p^v xx^* p^v)) = \tau(x^*x) + \tau(x^*(1 - p^v)x).$$

Therefore  $\|x\|_v = \|(2 - p^v)^{1/2}x\|_2$ , and thus

$$\|x\|_2 \leq \|x\|_v \leq \sqrt{2}\|x\|_2. \quad (2.4)$$

In [11], Mata and Recht introduced the classifying connection of a homogeneous reductive space. One of the main properties of this connection is that it has trivial torsion. The geodesics of this connection are computed explicitly in [11]. The unique curve  $\delta$  in  $\mathcal{I}_p$  with  $\delta(0) = v_0$  and  $\dot{\delta}(0) = xv$  is given by

$$\delta(t) = e^{t\kappa_v(x)}v.$$

Denote by  $P_v$  the map

$$P_v : M_{ah} \rightarrow M_{ah}, \quad P_v = \kappa_v \circ \ell_v.$$

Note that  $P_v$  takes values in  $\mathcal{H}_v$ , and since  $\ell_v \circ \kappa_v$  restricted to  $(T\mathcal{I}_p)_v$  is the identity map,  $P_v$  is an idempotent. Explicitly

$$P_v(x) = xp^v - p^v x^* - p^v xp^v = 2xp^v - p^v xp^v.$$

This idempotent  $P_v$  is in fact the orthogonal projection onto  $\mathcal{H}_v$ , with respect to the trace inner product: if  $x, y \in M_{ah}$ ,

$$\tau(y^* P_v(x)) = \tau(y^* 2xp^v - y^* p^v xp^v) = -\tau((2p^v y - p^v yp^v)x) = \tau(P_v(y)^* x).$$

The classifying connection  $\nabla^c$  is given as follows. Suppose that  $x, y$  are tangent vector fields in  $\mathcal{I}_p$ , then  $\nabla_x^c(y)$  is characterized by the value  $\kappa_v(\nabla_x^c(y))$  at each point  $v$ :

$$\kappa_v(\nabla_x^c(y)) = P_v(x(y)), \quad (2.5)$$

where  $x(y)$  denotes the derivative of  $y$  along  $x$ .

We shall prove next that this connection is the Levi-Civita connection of the metric (2.2) introduced above. By this we mean, that it is symmetric (torsion free) and compatible with the metric.

**Lemma 2.1** *The classifying connection  $\nabla^c$  is the Levi-Civita connection of the metric  $\langle \cdot, \cdot \rangle_v$  in  $\mathcal{I}_p$ .*

**Proof** It was proven in [11] that it is symmetric, let us show that it is compatible with the metric. Let  $x(t), y(t)$  be two tangent fields along the curve  $\nu(t)$  in  $\mathcal{I}_p$ . Then

$$\langle \frac{D^c x}{dt}, y \rangle_\nu = \tau(\kappa_\nu(y)^* P_\nu(\kappa_\nu(x))).$$

Note that since  $P_v$  is orthogonal with respect to the trace inner product, and projects onto  $\mathcal{H}_v$ , it follows that

$$\tau(\kappa_\nu(y)^* P_\nu(\kappa_\nu(x))) = \tau(\kappa_\nu(y)^* \dot{\kappa}_\nu(x)),$$

and accordingly for the term  $\langle x, \frac{D^c y}{dt} \rangle_\nu$ . Then

$$\begin{aligned} \langle \frac{D^c x}{dt}, y \rangle_\nu + \langle x, \frac{D^c y}{dt} \rangle_\nu &= \tau(\kappa_\nu(y)^* \dot{\kappa}_\nu(x)) + \tau(\kappa_\nu(y)^* \kappa_\nu(x)) \\ &= \frac{d}{dt}(\tau(\kappa_\nu(y)^* \kappa_\nu(x))) = \frac{d}{dt}(\langle x, y \rangle_\nu). \end{aligned}$$

□

### 3 Minimality of geodesics

If  $\nu$  is a smooth curve in  $\mathcal{I}_p$  with  $\nu(0) = v$ , there is a unique smooth curve  $\gamma$  in  $U_M$  with the following properties

1. The curve  $\gamma$  lifts  $\nu$ :  $\gamma(t)v = \nu(t)$ .
2.  $\gamma(0) = 1$ .
3.  $\gamma^* \dot{\gamma} \in \mathcal{H}_\nu$ .

This curve  $\gamma$  is usually called the *horizontal lifting* of  $\nu$ , and is also characterized as the unique solution of the following linear differential equation

$$\begin{cases} \dot{\gamma} &= \kappa_\nu(\dot{\nu})\gamma \\ \gamma(0) &= 1 \end{cases} \quad (3.6)$$

These are standard facts from the theory of homogeneous reductive spaces [11].

We shall need the following facts concerning the geometric structure of the unitary group  $U_M$ , which we take from [1] and [6]. The first states that if we measure the length of a smooth curve  $\gamma(t)$ ,  $t \in [a, b]$ , of unitaries using the  $\|\cdot\|_2$  induced by  $\tau$ , i.e.

$$L_2(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_2 dt,$$

then the curves of the form  $\delta(t) = ue^{tx}$ , with  $x \in M_{ah}$  have minimal length along their paths for  $t \in [0, 1]$ , provided that  $\|x\| \leq \pi$ . If  $\|x\| < \pi$ , the geodesic  $\delta$  is unique having this property. Note the fact that the condition is given on the *operator* norm of  $x$ , but the length is measured in the 2-norm. These norms are not equivalent, so this result is a weak form of a Hopf-Rinow theorem. We remark  $U_M$  is not a Hilbert-Riemann manifold with the trace inner product. See [1] for the details.

A straightforward consequence of the definition of the metric in  $\mathcal{I}_p$  is that the length of a curve  $\nu$  in  $\mathcal{I}_p$  coincides with the length of its horizontal lifting  $\gamma$ . Indeed,

$$L_2(\nu) = \int_0^1 \|\dot{\nu}\|_\nu dt = \int_0^1 \|\kappa_\nu(\dot{\nu})\|_2 dt = \int_0^1 \|\dot{\gamma}\gamma^*\|_2 dt = \int_0^1 \|\dot{\gamma}\|_2 dt = L_2(\gamma).$$

Thus we shall use the metric structure of the unitary group with the trace metric.

The next results on the metric geometry of  $U_M$  concern a variation formula for the energy functional and the local convexity property of the geodesic distance (i.e. the distance given by the minima of lengths of curves joining two given unitaries, which by the above cited result, are achieved by one parameter groups of unitaries). These facts were proved in [6].

In Theorem 2.1 of [6] it was shown that if  $F_2$  denotes the energy functional

$$F_2(\gamma) = \int_0^1 \|\dot{\gamma}\|_2^2 dt = \int_0^1 \tau(\dot{\gamma}^* \dot{\gamma}) dt,$$

for  $\gamma$  a piecewise smooth curve in  $U_M$ , and  $\gamma_s(t)$  is a smooth variation of  $\gamma$ , i.e.

$$\gamma_s(t) \in U_M, s \in (-r, r), \quad t \in [0, 1], \quad \gamma_0 = \gamma,$$

then the *first variation of the energy functional* is

$$\frac{1}{2} \frac{d}{ds} F_2(\gamma_s)|_{s=0} = \tau(x_0 y_0)|_{t=0}^{t=1} - \int_0^1 \tau\left(\frac{d}{dt}[x_0]y_0\right) dt, \quad (3.7)$$

where

$$x_s(t) = \gamma_s(t)^* \frac{d}{ds} \gamma_s(t) \quad \text{and} \quad y_s(t) = \gamma_s(t) \frac{d}{ds} \gamma_s(t).$$

The other result in [6] needed here, is the following lemma. Here  $d_k$  denotes the geodesic distance induced by the  $k$ -norm.

**Lemma 3.1** (Theorem 4.5 of [5]) *Let  $u_0, u_1, u_2 \in U_M$ , such that  $\|u_i - u_j\| < \sqrt{2 - \sqrt{2}} = r$ . Let  $\delta(t) = u_1 e^{tz}$  be the minimal geodesic joining  $u_1$  and  $u_2$ . Then  $f(s) = d_k(u_0, \delta(s))^k$  is a convex function ( $s \in [0, 1]$ ), for  $k$  an even integer.*

We shall use this result for the case  $k = 2$ .

Let us return now to  $\mathcal{I}_p$ .

**Remark 3.2** By virtue of the inverse function theorem, the exponential map

$$\exp_v : \mathcal{H}_v \rightarrow \mathcal{I}_p$$

is a local diffeomorphism near the origin. For  $r > 0$ , denote by  $B_r$  the ball of radius  $r$  centered at the origin in  $\mathcal{H}_v$ . Therefore there exists a number  $R_0$  such that  $\mathcal{E}_v = \exp_v(B_{R_0})$  is an open neighbourhood of  $v$  in  $\mathcal{I}_p$  (in the **norm** topology) and

$$\exp_v : B_{R_0} \rightarrow \mathcal{E}_v$$

is a diffeomorphism. In particular, for any element  $v_1 \in \mathcal{E}_v$ , there exists a unique geodesic  $\delta$  of  $\mathcal{I}_p$  inside  $\mathcal{E}_v$  with  $\delta(0) = v$  and  $\delta(1) = v_1$ . Namely,  $v_1 = \exp_v(z)$ , for  $z \in B_{R_0}$ , and  $\delta(t) = \exp_v(tz) = e^{tz}v$ .

In [2] a lower bound for  $R_0$  was given:  $R_0 \geq 0.0034$ . In what follows we assume that  $R_0 < \sqrt{2} - \sqrt{2}$ .

For any  $r \leq R_0$ , let us denote by  $\mathcal{E}_v(r) = \exp_v(B_r)$ , which is also an open neighbourhood of  $v \in \mathcal{I}_p$ .

Let us denote by  $L_\infty$  the length of curves, either of unitaries or partial isometries, measured with the operator norm  $\|\cdot\|$ : if  $\alpha(t)$ ,  $t \in [a, b]$ , is a curve (in  $U_M$  or  $\mathcal{I}_p$ )

$$L_\infty(\alpha) = \int_a^b \|\dot{\alpha}\| dt.$$

**Lemma 3.3** Let  $\nu(t)$ ,  $t \in [0, 1]$  be a smooth curve in  $\mathcal{I}_p$ , and let  $\gamma$  be its horizontal lifting. Then

$$L_\infty(\gamma) \leq \sqrt{2} L_\infty(\nu).$$

**Proof.** Note that

$$L_\infty(\gamma) = \int_0^1 \|\dot{\gamma}\| dt = \int_0^1 \|\kappa_\nu(\dot{\nu})\| dt.$$

Thus it suffices to show that for any tangent vector  $x \in (T\mathcal{I}_p)_v$ ,  $\|\kappa_v(x)\| \leq \sqrt{2}\|x\|$ . Recall that  $\kappa_v(x) = xv^* - vx^* - p^v xv^* = (1 - p^v)xv^* - vx^*$ . Since  $p^v$  is the range projection of  $v$ , it follows that the operators  $(1 - p^v)xv^*$  and  $vx^*$  have orthogonal ranges. Then, if  $\xi$  is a unit vector in any representation of  $M$ ,

$$\|\kappa_v(x)\xi\|^2 = \|(1 - p^v)xv^*\xi\|^2 + \|vx^*\xi\|^2$$

so that

$$\|\kappa_v(x)\|^2 \leq \|(1 - p^v)xv^*\|^2 + \|vx^*\|^2 \leq 2\|x\|^2.$$

□

**Lemma 3.4** Let  $\nu(t)$ ,  $t \in [0, 1]$  be a smooth curve in  $\mathcal{I}_p$  joining  $v$  and  $e^z v$ ,  $z^* = -z$ . Suppose that  $\|z\| < R_0/2$  and  $L_\infty(\nu) < R_0/2\sqrt{2}$ . Denote by  $\gamma$  the horizontal lifting of  $\nu$ . Then  $1$ ,  $e^z$  and  $\gamma(1)$  lie at (norm) distance less than  $\sqrt{2} - \sqrt{2}$ .

**Proof.** First note that from the assumption on the size of  $R_0$ , clearly  $\|z\| \leq \pi/4$ , and therefore

$$\|1 - e^z\| = \sqrt{2 - 2\cos(\|z\|)} < \sqrt{2} - \sqrt{2}.$$

Also it is clear that  $L_\infty(\gamma) \leq \sqrt{2}L_\infty(\nu) < R_0/2$ . Let  $d_\infty$  denote the geodesic distance in  $U_M$  given by the usual norm. It follows that  $d_\infty(1, \gamma(1)) < R_0/2$ . Therefore  $\gamma(1) = e^y$ , with  $y^* = -y$  and  $\|y\| < R_0/2$ . Then, similarly as above,

$$\|1 - \gamma(1)\| = \|1 - e^y\| = \sqrt{2 - 2\cos(\|y\|)} < \sqrt{2} - \sqrt{2}.$$



Finally

$$d_\infty(\gamma(1), e^z) \leq d_\infty(\gamma(1), 1) + d_\infty(1, e^z) \leq \|y\| + \|z\| < R_0,$$

Then there exists  $w \in M_{ah}$ ,  $\|w\| < R_0 < \pi/4$  such that  $\gamma(1) = e^z e^w$ , and then again

$$\|\gamma(1) - e^z\| = \|1 - e^w\| < \sqrt{2 - \sqrt{2}}.$$

□

The next result states that among curves in  $\mathcal{I}_p$  which are sufficiently short for the  $L_\infty$  functional, the geodesics have minimal  $L_2$  length.

**Lemma 3.5** *Let  $\nu$  be a smooth curve in  $\mathcal{I}_p$  such that  $L_\infty(\nu) < R_0/2\sqrt{2}$ . Then there exists a geodesic  $\delta$  with the same endpoints as  $\nu$ , such that*

$$L_2(\delta) \leq L_2(\nu).$$

**Proof.** Put  $v = \nu(0)$ . Note that  $L_\infty(\nu) < R_0/2\sqrt{2}$  implies in particular that  $\nu(1) \in \mathcal{E}_v(R_0/2)$ , and therefore there exists  $z \in \mathcal{H}_v$  with  $\|z\| < R_0/2$  such that  $\exp_v(z) = e^z v = \nu(1)$ . Let  $\gamma$  be the horizontal lifting of  $\nu$ . By the lemma above, the unitaries  $\gamma(1)$ ,  $e^z$  and 1 differ in norm less than  $\sqrt{2 - \sqrt{2}}$ . Let  $\mu(s) = e^z e^{sy}$  be the minimal geodesic (for the  $\|\cdot\|_2$  metric) of  $U_M$  joining  $\mu(0) = e^z$  and  $\mu(1) = e^z e^y = \gamma(1)$  (note that indeed  $\|y\| < \pi$ ).

Consider the map

$$f(s) = d_2(1, \mu(s))^2, \quad s \in [0, 1].$$

By (3.1),  $f(s)$  is convex. We claim that  $f'(0) = 0$ , so that  $f$  has an absolute minimum at  $s = 0$ .

Indeed, note that

$$\|\mu(s) - 1\| = \|1 - e^{sy}\| \leq \|1 - e^y\| = \|e^z - \gamma(1)\| < \sqrt{2 - \sqrt{2}} < 2.$$

Therefore the antihermitic logarithm

$$\log : \{u \in U_M : \|u - 1\| < 2\} \rightarrow \{x \in M_{ah} : \|x\| < \pi\}$$

is well defined and smooth. Let  $\epsilon_s(t) = e^{t \log(\mu(s))}$ . Note that  $\epsilon_s(t)$  is a smooth variation of  $\epsilon_0 = e^{tz}$ . Also note that at each  $s$  it consists of minimizing geodesics, because  $\|\log(\mu(s))\| < \pi$ . Then

$$f(s) = L_2(\gamma_s)^2 = \|\log(\mu(s))\|_2^2 = F_2(\epsilon_s).$$

Then  $f'(0)$  can be computed using the first variation formula (3.7). In our case

$$x_s = \epsilon_s^* \frac{d}{dt} \epsilon_s(t) = \log(\mu(s))$$

is independent of  $t$ , and therefore (3.7) reduces to

$$f'(0) = 2\{\tau(zy_0(1)) - \tau(zy_0(0))\}.$$

Note that  $\gamma(1)$  and  $e^z$  verify that  $\gamma(1)v = e^z v = \nu(1)$ , i.e.  $e^{-z}\gamma(1) \in V_v$ . Also note that  $V_v \subset U_M$  is geodesically convex: the minimal geodesic joining 1 and  $e^{-z}\gamma(1)$ , namely  $e^{sy}$ , lies in  $V_v$ . Therefore  $y \in \mathcal{V}_v$ .

At  $t = 0$ ,  $\epsilon_s(0) = 1$  for all  $s$ , therefore  $y_s(0) = 0$ , and in then  $\tau(zy_0(0)) = 0$ . At  $t = 1$ ,  $\epsilon_s(1) = \mu(s)$ , so that  $y_s(1) = \mu^*(s)\dot{\mu}(s)$  and  $y_0(1) = y$ . Then

$$\tau(zy_0(1)) = \tau(zy) = 0$$

because  $z \in \mathcal{H}_v$  and  $y \in \mathcal{V}_v$  are  $\tau$ -orthogonal subspaces.

Thus  $f(0) \leq f(1)$ , and therefore

$$L_2(\epsilon_0) = d_g(1, e^z) \leq d_g(1, \gamma(1)) = L_2(\gamma) :$$

Since  $\epsilon_0$  and  $\gamma$  are the horizontal liftings of  $\delta$  and  $\nu$ , respectively, we have that

$$L_2(\delta) \leq L_2(\nu).$$

□

We call a piecewise smooth curve a geodesic polygonal if it is a continuous path in  $\mathcal{I}_p$ , and consists of geodesic paths glued together. The next result states that given any piecewise smooth curve, there is a geodesic polygonal joining the same endpoints which is shorter than the original curve.

**Proposition 3.6** *Let  $\nu(t)$ ,  $t \in [0, 1]$  be a piecewise smooth curve, then there is a geodesic polygonal  $\rho$  such that  $\rho(0) = \nu(0)$ ,  $\rho(1) = \nu(1)$  and*

$$L_2(\rho) \leq L_2(\nu).$$

**Proof.** Clearly it suffices to consider the case when  $\nu$  is smooth. Then there exists a partition  $t_0 = 0 < t_1 < \dots < t_n = 1$  of the unit interval such that for all  $i = 1, \dots, n$ ,  $\|\nu(t_i) - \nu(t_{i-1})\| < R_0/2$ . Thus each curve  $\nu|_{[t_{i-1}, t_i]}$  satisfies the hypothesis of the preceding lemma, therefore there exists a geodesic  $\rho_i$  with the same endpoints as  $\nu|_{[t_{i-1}, t_i]}$  with

$$L_2(\rho_i) \leq L_2(\nu|_{[t_{i-1}, t_i]}).$$

Clearly the curve  $\rho$  which consists of gluing together the paths  $\rho_i$  is a continuous polygonal which is shorter than  $\nu$ . □

**Theorem 3.7** *Given any point  $v' \in \mathcal{E}_v(R_0/4\sqrt{2})$ , there exists a unique geodesic  $\delta(t) = e^{tz}v$  with  $\delta(1) = e^z v = v'$  which is shorter than any other piecewise smooth curve  $\nu \subset \mathcal{E}_v(R_0/4\sqrt{2})$  joining  $v$  and  $v'$ .*

**Proof.** Clearly, by remark (3.2), there exists  $z \in \mathcal{H}_v$  such that  $e^z v = v'$ . As in the proof of the preceding proposition, we may obtain a geodesic polygonal with vertices  $v_0 = v, v_1, \dots, v_n = v'$  in  $\mathcal{E}_v(R_0/4\sqrt{2})$ , with edges of  $L_\infty$ -length less than  $R_0/4\sqrt{2}$ , which is shorter than  $\nu$  for the  $L_2$  functional. Let us denote by  $\epsilon_i$  the edges of this polygonal (joining  $v_{i-1}$  and  $v_i$ ), and by  $\epsilon_i \# \epsilon_{i+1}$  the path formed by adjoining two consecutive edges. Note that

$$L_\infty(\epsilon_1 \# \epsilon_2) \leq L_\infty(\epsilon_1) + L_\infty(\epsilon_2) < R_0/2\sqrt{2}.$$

Therefore, by lemma (3.5), the geodesic polygonal which consists of replacing  $\epsilon_1 \# \epsilon_2$  by the minimal geodesic  $\epsilon'_1$  joining  $v_0$  and  $v_2$  in the original polygonal, is shorter than this polygonal for the  $L_2$  functional. Note that the new edge  $\epsilon'_1$  lies inside  $\mathcal{E}_v(R_0/4\sqrt{2})$ . Proceeding inductively, one arrives at the desired result. □

We say that a curve  $\nu(t)$ ,  $t \in [0, 1]$ , is minimal along its path if for any  $0 \leq t_0 < t_1 \leq 1$ , the curve  $\nu|_{[t_0, t_1]}$  has minimal length among all curves in  $\mathcal{I}_p$  joining  $\nu(t_0)$  and  $\nu(t_1)$ . Note that we have proved that the geodesics of the connection that are minimal, are in fact minimal along their paths.

Next let us show that the geodesics of the connection are the only possible curves which have minimal length.

**Theorem 3.8** *Suppose that  $\nu$  is a piecewise smooth curve which has minimal  $L_2$ -length along smooth curves in  $\mathcal{I}_p$  joining the same endpoints. Then  $\nu$  is a geodesic of the linear connection.*

**Proof.** By the previous proposition, there exists a geodesic polygonal  $\epsilon$  with edges  $\epsilon_i$ ,  $i = 1, \dots, n$ , which is not longer than  $\nu$ . Then  $\epsilon$  is also minimal. Moreover, by construction, the elements  $t_0, t_1, \dots, t_n$  of the partition provide points  $\nu(t_i)$  which lie both in  $\nu$  and  $\epsilon$ . We claim that  $\epsilon$  is smooth, i.e. a geodesic. This proves our result, since the partition can be arbitrarily refined to contain as many (finite) points in common between  $\nu$  and  $\epsilon$ , and the smoothness of  $\epsilon$  proves that all these polygonals are in fact the same geodesic, and coincides with  $\nu$ .

We assume that  $\epsilon$  is not smooth to arrive to a contradiction. Namely, suppose that there are points  $v_i$  occurring at instants  $t_i$  of the polygonal  $\epsilon$  where  $\dot{\epsilon}(t_i^-) \neq \dot{\epsilon}(t_i^+)$ . The edges  $\epsilon_i$  and  $\epsilon_{i+1}$  of  $\epsilon$  at the vertex  $v_i$  can be parametrized  $\epsilon_i(t) = e^{tz_i^+} v_i$  and  $\epsilon_{i+1}(t) = e^{tz_i^-} v_i e^{-tz^-}$ , then the jumps of the derivative of  $\epsilon$  are

$$0 \neq \Delta \dot{\epsilon}_i = \dot{\epsilon}_{i+1}(0^+) - \dot{\epsilon}_i(1^-) = z_i^+ v_i - z_i^- v_i$$

is a nonzero vector in  $\mathcal{H}_q$ . We may choose a variation  $\gamma_s$  of  $\epsilon = \gamma_0$  which is constantly identical to  $\epsilon$  except in small neighbourhoods of  $q_i$  (so that they do not overlap), and such that the variation field  $V(t) = \frac{\partial}{\partial s} |_{s=0} \gamma_s$  equals these jumps (which are tangent vectors) at  $t = t_i$ , namely  $V(t_i) = \Delta \dot{\epsilon}_i$ .

According to the classic first variation formula in a Riemannian manifold (cf. [9] for example),

$$\frac{d}{ds} |_{s=0} L_2(\gamma_s) = - \int_0^1 \langle V, D_t^c \dot{\gamma} \rangle dt - \sum \langle V(t_i), \Delta_i \dot{\gamma} \rangle.$$

Since  $\epsilon$  consists of piecewise geodesics, and is also a critical point of the length distance (it is minimizing for the 2-metric) we obtain

$$0 = \frac{d}{ds} |_{s=0} L_2(\gamma_s) = \sum_i \langle z_i^+ v_i - z_i^- v_i, z_i^+ v_i - z_i^- v_i \rangle^{1/2} = \sum_i \|z_i^+ - z_i^-\|_2,$$

which is a contradiction. □

## 4 Completeness of the Riemannian metric

In this section we shall prove that the geodesic distance  $d_g$  induced by the Riemannian metric is complete. An interesting fact, given that the tangent spaces of  $\mathcal{I}_p$  are themselves non complete. This fact is related to the completeness of the space

$$\mathcal{P} = \{q \in M : q^2 = q^* = q\}$$

of selfadjoint projections of  $M$ , also called Grassmannian of  $M$ . The Grassmannian of a  $C^*$ -algebra is a well studied space, there are several papers considering the geometric aspects of this set: [?], [7], [12]. In [5] we considered the Grassmannian of a finite von Neumann algebra, and endowed their tangent spaces with the uncomplete Riemannian metric given by the 2-norm, in the same spirit as in the present paper. Let us list a few properties obtained there:

1. The Levi-Civita connection of the  $\|\cdot\|_2$  at every tangent space of  $\mathcal{P}$  is the reductive connection introduced in [?]. The geodesics of this connection are the curves of the form

$$\rho(t) = e^{tx} q e^{-tx},$$

where  $x^* = -x$  is co-diagonal with respect to  $q$ :  $x = xq + qx$ .

2. Given two projections  $q_1, q_2$ , there is always a geodesic joining them.

3. The geodesic above can be chosen minimal. Namely, a minimal geodesic verifies that  $\|x\| \leq \pi/2$ . If  $\|q_1 - q_2\| < 1$  (in general one has  $\|q_1 - q_2\| \leq 1$ ), then the minimal geodesic is unique.
4. The geodesic distance (i.e. the metric given by the infima of lengths of curves in  $\mathcal{P}$  joining the giving endpoints) is complete, and equivalent to the norm  $\|\cdot\|_2$ .

We shall use these facts to prove completeness of  $(\mathcal{I}_p, d_g)$ . There is one more property of  $\mathcal{P}$  which shall be needed, and is not contained in [5]. It is the following sequential lifting property for the strong topology.

**Proposition 4.1** *Let  $q_n$  be a sequence in  $\mathcal{P}$  which converges strongly to  $q$ . Then there exist unitaries  $u_n \in U_M$  such that*

$$u_n q u_n^* = q_n \quad \text{and} \quad u_n \rightarrow 1 \text{ strongly.}$$

**Proof.** Since the norm  $\|\cdot\|_2$  metrizes the strong topology in  $\mathcal{P}$ , and it is equivalent to the geodesic distance, there exist minimal geodesics  $\rho_n(t) = e^{tx_n} q e^{-tx_n}$  joining  $q$  to  $q_n$ , such that

$$L_2(\rho_n) = \|x_n q - q x_n\|_2 \rightarrow 0.$$

Note that since  $x_n = x_n q + q x_n$ , then  $\|x_n q - q x_n\|_2 = \|x_n\|_2$ . Moreover, the norms  $\|x_n\|$  are bounded by  $\pi/2$ . In [1], it was proven that if  $\|x_n\| \leq \pi$ , then  $\|x_n\|_2 \rightarrow 0$  if and only if  $e^{x_n} \rightarrow 1$  in  $U_M$  strongly.  $\square$

The link between these facts and the topology of  $\mathcal{I}_p$  is the map

$$\Phi : \mathcal{I}_p \rightarrow \mathcal{P}, \quad \Phi(v) = vv^* = p^v \tag{4.8}$$

which assigns to each partial isometry its final projection. Since the algebra is finite, it is clearly strongly continuous. Also, when regarded between the smooth structures (given by the usual norm) it is  $C^\infty$ . For our purposes, the relevant property of  $\Phi$  is the following:

**Proposition 4.2** *The differential  $d\Phi_v$  of  $\Phi$  is contractive, when the tangent space of  $\mathcal{I}_p$  is endowed with the Riemannian metric  $\|\cdot\|_v$ , and the tangent space of  $\mathcal{P}$  is endowed with the norm  $\|\cdot\|_2$ .*

**Proof.** Note that

$$d\Phi_v(x) = xv^* + vx^*, \quad x \in (T\mathcal{I}_p)_v.$$

Then

$$\|d\Phi_v(x)\|_2^2 = \|xv^* + vx^*\|_2^2 = \tau(xp x^*) + \tau(xv^* x v^*) + \tau(vx^* v x^*) + \tau(vx^* x v^*).$$

Tangent vectors  $x$  at  $v$  satisfy  $x^*v + v^*x = 0$  (for instance, differentiate the equation  $v^*v = p$ ). Also, since  $\mathcal{I}_p \subset Mp$ ,  $xp = x$ . Therefore

$$\|d\Phi_v(x)\|_2^2 = 2\tau(xx^*) - 2\tau(p^v x x^* p^v) \leq 2\tau(xx^*) - \tau(p^v x x^* p^v) = \|x\|_v^2.$$

$\square$

Our main result of this section follows:

**Theorem 4.3** *The metric space  $(\mathcal{I}_p, d_g)$  is complete.*

**Proof.** Let  $v_n$  be a Cauchy sequence in  $\mathcal{I}_p$  for the geodesic distance. Then, by the above proposition,  $q_n = v_n v_n^*$  is a Cauchy sequence for the geodesic distance in  $\mathcal{P}$ . Therefore there exists  $q \in \mathcal{P}$  such that  $q_n \rightarrow q$  strongly. By the sequential lifting property there exist  $x_n^* = -x_n \in M$  with  $\|x_n\| \leq \pi$  such that  $e^{x_n} q e^{-x_n} = q_n$  and  $\|x_n\|_2 \rightarrow 0$ .

We claim that  $e^{-x_n}v_n$  is also a Cauchy sequence in  $\mathcal{I}_p$  for the geodesic distance. Indeed,

$$d_g(e^{-x_n}v_n, e^{-x_m}v_m) \leq d_g(e^{-x_n}v_n, v_n) + d_g(v_n, v_m) + d_g(v_m, e^{-x_m}v_m).$$

Note that  $e^{-x_n}v_n$  can be joined to  $v_n$  by means of the curve  $\nu_n(t) = e^{-tx_n}v_n$ , whose length is

$$L_2(\nu_n) = \int_0^1 \|\dot{\nu}_n\|_{\nu_n} dt = \sqrt{2} \int_0^1 \|\dot{\nu}_n\|_2 dt = \sqrt{2} \|x_n v_n\|_2 \leq \sqrt{2} \|x_n\|_2.$$

It follows that  $d_g(e^{-x_n}v_n, v_n) \leq L_2(\nu_n) \rightarrow 0$ . Analogously,  $d_g(e^{-x_m}v_m, v_m) \rightarrow 0$ .

Note that  $e^{-x_n}v_n$  are partial isometries with (initial space  $p$  and) final space equal to  $q$ :

$$e^{-x_n}v_n(e^{-x_n}v_n)^* = e^{-x_n}v_n v_n^* e^{x_n} = e^{-x_n}q_n e^{x_n} = q.$$

Since  $q = upu^*$  for some unitary  $u$ , this subset of  $\mathcal{I}_p$  can be made homeomorphic with the set  $\mathcal{I}_p^p \subset \mathcal{I}_p$  of partial isometries with final and initial space equal to  $p$ , by means of

$$v \mapsto u^*v.$$

We may change the metric in  $\mathcal{I}_p^p$  considering the 2-norm at every tangent space. By (2.4), the new metric is equivalent to the former, in particular, they have the same Cauchy sequences. Now this set  $\mathcal{I}_p^p$ , clearly identifies isometrically the unitary group of  $pMp$ . In [1] it was shown that this space is complete with the geodesic distance. Completeness of  $\mathcal{I}_p$  follows.  $\square$

A similar type of argument shows that the mapping

$$L_v : U_M \rightarrow \mathcal{I}_p, \quad L_v(u) = uv$$

has also the sequential lifting property for the strong topology. Recall that in the norm topology it is a submersion, and therefore a fibration.

**Proposition 4.4** *Let  $v_n$  be a sequence in  $\mathcal{I}_p$  converging strongly to  $v$ . Then there exist unitaries  $u_n \in U_M$  such that  $u_n v = v_n$  and  $u_n \rightarrow 1$  strongly. In particular,  $\mathcal{I}_p$  is homeomorphic to  $U_M/V_v$ , when all the spaces involved are considered with the strong operator topology.*

**Proof.** The action of  $U_M$  on  $\mathcal{I}_p$  is transitive, the maps  $\mathcal{I}_p \ni v \mapsto uv \in \mathcal{I}_p$  are homeomorphisms for the strong topology, therefore we may suppose without loss of generality that  $v = p$ . Since  $M$  is finite and  $v_n$  are uniformly bounded,  $q_n = v_n v_n^* \rightarrow p$  strongly. By the analogous result above for  $\mathcal{P}$ , there exist unitaries  $w_n$  in  $M$  which converge strongly to 1, such that  $q_n = w_n p w_n^*$ . Then one can choose  $u_n = v_n + w_n(1 - p)$ . Indeed note that  $u_n \rightarrow 1$  strongly, and  $u_n p = v_n p = v_n$ . It remains to show that they are unitary elements:

$$u_n u_n^* = v_n v_n^* + w_n(1 - p)w_n^* + v_n(1 - p)w_n^* + w_n(1 - p)v_n^* = q_n + 1 - q_n = 1,$$

because  $v_n(1 - p)w_n^* = 0 = (v_n(1 - p)w_n^*)^* = w_n(1 - p)v_n^*$ .

The map  $L_v : U_M \rightarrow \mathcal{I}_p$  which is continuous in the strong operator topology, induces a continuous bijection

$$\bar{L}_v : U_M/V_v \rightarrow \mathcal{I}_p,$$

where  $\mathcal{I}_p$  is considered with the strong operator topology. The sequential lifting property implies that it is open, i.e. a homeomorphism. Indeed, if  $v_n \rightarrow v$  strongly, then there exist unitaries  $u_n$  such that  $u_n v = v_n$  and  $u_n \rightarrow 1$  strongly. Therefore  $[u_n] \rightarrow [1]$  in  $U_M/V_v$ .  $\square$

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