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**“On the logarithmic derivative of the  $\zeta$ -determinant”**

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# On the logarithmic derivative of the $\zeta$ -determinant <sup>\*†</sup>

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## Abstract

A formula for the derivative of the logarithm of the  $\zeta$ -determinant of the quotient of two elliptic pseudodifferential operators with the same boundary condition, acting between the fibers of a vector bundle over a  $n$ -dimensional manifold  $M$  with boundary  $X$ , is here presented.

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# 1 Introduction

Given a trace class operator  $A$  acting on a Hilbert space, the Fredholm determinant of the operator  $L = I - A$  is defined by

$$\det_1 L = \prod_{j=1}^{\infty} (1 - \lambda_j(A)), \quad (1)$$

where  $I$  is the identity operator and the numbers  $\lambda_j(A)$  are the eigenvalues of  $A$ , repeated the times indicated by their corresponding multiplicities.

It is a very known fact the necessity of this concept in various areas of mathematics, as differential geometry [5], and those of physics, for instance in the construction of quantum theories by means of functional integration ([16], [6], [7], [2], etc.), in where the calculus of determinants of quotients of some elliptic differential operators recovers a special interest.

In [5] R. Forman has studied some Fredholm determinant properties of  $L$  and the quotient of regularization of the determinants of two differential operators  $D_0$  and  $D_1$  by the Riemann  $\zeta$ -function method, when  $L = D_0 D_1^{-1} = I - A$  and  $A$  belongs to the trace class operators. This type of determinant regularization procedure is called the  $\zeta$ -determinant regularization and is denoted by  $Det_{\zeta}$ .

In general, for an operator  $L$  acting on a Hilbert space  $H$  the notions of Fredholm determinant and the  $\zeta$ -determinant have no sense. On the other way, in several occasions, the interest is focalized on the quotient of the determinants of the operators instead of each determinant individually. On this line the works [6] and [7] fit in perfectly. It is shown in such papers that the quotient between the  $\zeta$ -determinants of two elliptic operators  $A + \epsilon A_1$  and  $A$ , defined on a compact differential manifold without boundary, is given by

$$\frac{Det_{\zeta}(A + \epsilon A_1)}{Det_{\zeta}(A)} = \exp \left\{ \epsilon \left. \frac{d}{ds} \right|_{s=0} [s.Tr(A^{-s-1} A_1)] + O(\epsilon^2) \right\}, \quad (2)$$

where  $A$  is pseudodifferential of positive order and  $A_1$  is a differential operator with  $order(A_1) < order(A)$ .

Another version about the derivative of the logarithm of the  $\zeta$ -determinant with respect to a parameter is presented in [5] where it is established that

$$\begin{aligned} \frac{d}{dt} \log \text{Det}_\zeta L_{tB} &= \frac{d}{ds} \text{Tr} \left[ s \cdot \left( \frac{d}{dt} L_{tB} \right) \cdot L_{tB}^{-s-1} \right] \Big|_{s=0} \\ &= \frac{d}{dt} \log \det_1 (L_{tB} \cdot L_{0B}^{-1}), \end{aligned} \quad (3)$$

for a quotient of elliptic differential operators belonging to a monoparametric family  $L_t$ , all with identical principal symbol (and, hence, with the same order) and the same elliptic boundary condition  $B$  for each member  $L_t$  of the family. For the veracity of this formula R. Forman requires the restrictive hypothesis that  $\left(\frac{d}{dt} L_{tB}\right) L_{tB}^{-1}$  is a trace class operator for all  $t$ . It will be shown that this restriction can be removed. So, one aim of this paper is to extend Forman's result to the quotient of two classical elliptic pseudodifferential operators.

The paper has three sections. Next part of the present section is devoted to expose the principal results. Same basic concepts, notation and definitions as the  $\zeta$ -determinant regularization method and some differential properties of the Fredholm determinant are recalled in section two. In the last section the extended results to pseudodifferential operators are proved.

## 1.1 Main results

Now we are in condition to present our principal statements about the logarithm of the  $\zeta$ -determinant of the quotient of two (classical) elliptic pseudodifferential operators. The first theorem refers to the version of operators defined over a compact manifold without boundary whereas the second treats the case of two elliptic boundary problems, both with the same boundary elliptic condition.

### Theorem 1.1.

*Let  $\Omega$  be an open subset of the complex plane and let  $z(t) : [0, 1] \longrightarrow \Omega$  be a differentiable curve. Over a compact,  $n$ -dimensional, differential manifold  $M$  without boundary define the  $z$ -analytic family  $\{L_z\}_{z \in \Omega}$  of elliptic, invertible, pseudodifferential operators, having order  $m > 0$ . For simplicity, let it be denoted  $L_t = L_{z(t)}$ .*

It is supposed that all the operators of the family have the same principal symbol, which has a cone of minimum growth rays, that is, a cone of rays on  $\mathbb{C}$  in which the principal symbol does not have any eigenvalue.

Then, for all  $t \in [0, 1]$  it is satisfied

$$\frac{d}{dt} \ln \text{Det}_\zeta L_t = \frac{d}{ds} \Big|_{s=0} \text{Tr} \left[ s \cdot \left( \frac{d}{dt} L_t \right) \cdot L_t^{-s-1} \right], \quad (4)$$

being the r.h.s. of this equality the “finite part” at  $s = 0$  of the analytic extension of  $\text{Tr} \left[ \left( \frac{d}{dt} L_t \right) \cdot L_t^{-s-1} \right]$ .

**Theorem 1.2.**

Let  $\Omega$  be an open subset of the complex plane and let  $z(t) : [0, 1] \longrightarrow \Omega$  be a differentiable curve. Over a compact,  $n$ -dimensional, differential manifold  $M$  with boundary  $X$  define the  $z$ -analytic family  $\{L_z\}_{z \in \Omega}$  of elliptic, invertible, pseudodifferential operators, having order  $m > 0$ . Let  $B$  be the same elliptic boundary condition for each  $L_z$ . Let denote with  $L_t$  the elliptic problem  $(L_{z(t)}, B)$ , for all  $t \in [0, 1]$ .

It is supposed that all the operators of the family have the same principal symbol, which has a cone of minimum growth rays.

Then, for all  $t \in [0, 1]$  it is satisfied

$$\frac{d}{dt} \ln \text{Det}_\zeta L_t = \frac{d}{ds} \Big|_{s=0} \text{Tr} \left[ s \cdot \left( \frac{d}{dt} L_t \right) \cdot L_t^{-s-1} \right], \quad (5)$$

being the r.h.s. of this equality the “finite part” at  $s = 0$  of the analytic extension of  $\text{Tr} \left[ \left( \frac{d}{dt} L_t \right) \cdot L_t^{-s-1} \right]$ .

Next, the corresponding integrated version will be enunciated. In order to deduce the first corollary it is enough to take the exponential function after integrating from 0 to  $t_o$  in equations (4) or (5) of the previous theorems.

**Corollary 1.3.** *(Integrated version)*

Under the hypotheses of theorem 1.1 or theorem 1.2, it is true that

$$\frac{\text{Det}_\zeta L_{t_o}}{\text{Det}_\zeta L_0} = \exp \left\{ \int_0^{t_o} \frac{d}{ds} \Big|_{s=0} \left\{ s \cdot \text{Tr} \left[ \frac{d}{dt} (L_t) \cdot L_t^{-s-1} \right] \right\} dt \right\}.$$

**Corollary 1.4.** (*Logarithmic derivative trace class case*)

Under the hypotheses of theorem 1.1 or theorem 1.2, if besides  $\left(\frac{d}{dt}L_t\right) \cdot L_t^{-1}$  is a trace class operator for all  $t$ , it is valid that

$$\begin{aligned}\frac{d}{dt} \log \text{Det}_\zeta L_t &= \frac{d}{ds} \text{Tr} \left[ s \cdot \left( \frac{d}{dt} L_t \right) \cdot L_t^{-s-1} \right] \Big|_{s=0} \\ &= \frac{d}{dt} \log \det_1 (L_t \cdot L_0^{-1}),\end{aligned}\tag{6}$$

and also

$$\frac{\text{Det}_\zeta L_{t_0}}{\text{Det}_\zeta L_0} = \det_1 (L_{t_0} \cdot L_0^{-1}).$$

**Remark 1.5.** It should be noted that the previous corollary is just one of the results established in [5].

## 2 Fredholm determinant and $\zeta$ -determinant regularization method

### 2.1 Basic concepts, technical explanations and notation

As usual,  $\mathbb{N}$  will denote the set of the positive integers,  $\mathbb{R}$  the set of real numbers and  $\mathbb{C}$  the set of complex numbers. If  $\omega \in \mathbb{C}$ , its real and complex parts are denoted by  $\text{Re}(\omega)$  and  $\text{Im}(\omega)$ , respectively. The greek letters  $\alpha, \beta, \dots$  are used for multi-indexes of numbers in  $\mathbb{N}$ ; in this way

$$\begin{aligned}\alpha &= (\alpha_1, \alpha_2, \dots, \alpha_n) \quad , \quad \beta = (\beta_1, \beta_2, \dots, \beta_n) \\ \alpha + \beta &= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n) \\ \alpha! &= \alpha_1! \alpha_2! \dots \alpha_n! \quad , \quad |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n.\end{aligned}$$

The letters  $x, y, \xi$  denote points in the Euclidean space  $\mathbb{R}^n$ . Then,

$$\begin{aligned}x &= (x_1, x_2, \dots, x_n) \quad , \quad y = (y_1, y_2, \dots, y_n) \\ \langle x, y \rangle &= x_1 y_1 + x_2 y_2 + \dots + x_n y_n \\ x^\alpha &= x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \\ \partial_x^\alpha &= \left( \frac{\partial}{\partial x} \right)^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \left( \frac{\partial}{\partial x_2} \right)^{\alpha_2} \dots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}.\end{aligned}$$

Let  $M$  be a differential manifold equipped by a measure  $\mu$ . The space of all the complex valued functions defined over  $M$  having derivatives of every order will be denoted by

$$\mathcal{C}^\infty(M) = \{f : M \longrightarrow \mathbb{C} / f \text{ is infinitely differentiable}\}.$$

In general,  $H$  will be understood a Hilbert space and the set of all the linear and continuous operators  $T : H \longrightarrow H$  will be denoted  $\mathcal{L}(H)$ . In particular, the Hilbert space of the square integrable functions  $f : M \longrightarrow \mathbb{C}$  will be denoted by  $H = L^2(M)$ .

The letters  $L, L_1, L_t$ , etc. indicate differential or pseudodifferential operators, and  $A, B$ , etc. boundary conditions. For the vector bundles over  $M$  it will be written  $(E, M, \pi_E)$  and  $(F, M, \pi_F)$ .

A classical pseudodifferential operator  $L$  of order  $m$  defined from the  $\mathcal{C}^\infty$  sections of the vector bundle  $(E, M, \pi_E)$  to the  $\mathcal{C}^\infty$  sections of another vector bundle  $(F, M, \pi_F)$  is a linear operator that, for each local patch  $(\mathcal{O}, \varphi)$  of  $M$  and for each local section  $f$  over  $\mathcal{O}$ , can be expressed

$$L f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle \varphi(x), \xi \rangle} \sigma(L)(\varphi(x), \xi) \widehat{f \circ \varphi^{-1}}(\xi) d\xi,$$

where  $\widehat{g}(\xi)$  indicates the Fourier transform of the function  $g$ , and  $\sigma(L)(y, \xi)$  is the so called (full) symbol of  $L$  and is a  $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  function satisfying

$$|\partial_y^\alpha \partial_\xi^\beta \sigma(L)(y, \xi)| \leq C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|},$$

for all pair of multi-indexes  $\alpha, \beta$  and some constant  $C$  only depending on them.

In the case in which the (full) symbol  $\sigma(L)(x, \xi)$  of  $L$  admits an asymptotic expansion

$$\sum_{j \geq 0} a_{m-j}(x, \xi),$$

being  $a_{m-j}(x, \xi)$  the  $\mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  functions which are homogeneous in  $|\xi| \geq 1$  of degree  $m - j$ , we say that the operator  $L$  belongs to the class  $I_h^m(M)$ . The principal symbol of  $L$ , denoted by  $\sigma_0(L)$ , is the function  $a_m(x, \xi)$  of the last asymptotic expansion of the symbol.

The composition of two operators  $L_1$  and  $L_2$  belonging to  $I_h^{m_1}(M)$  and  $I_h^{m_2}(M)$ , respectively, is another classical pseudodifferential operator in the class  $I_h^{m_1+m_2}(M)$ . Its (full)

symbol is given by the expression (cf. [3], [9]).

$$\sigma(L_1 L_2) = \sigma(L_1) \circ \sigma(L_2) \sim \sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!} (\partial_{\xi}^{\alpha} p) (\partial_x^{\alpha} q),$$

with  $p = \sigma(L_1)$  and  $q = \sigma(L_2)$ . In particular, for the principal symbol we have the simple relationship  $\sigma_0(L_1 L_2) = \sigma_0(L_1) \sigma_0(L_2)$ .

A  $k \times k$  matrix of pseudodifferential operators  $L \in I_h^m(M)$  is called (uniformly) elliptic if its principal symbol  $\sigma_0(L)$  satisfies

$$|\det \sigma_0(L)(x, \xi)| \geq C |\xi|^{mk}, \quad \text{for } |\xi| > N \text{ and } C > 0.$$

When  $M$  is supposed a differential manifold with boundary  $X$ , for each (classical) elliptic pseudodifferential operator  $L$  acting between the fibers of two vector bundles over  $M$ , there exists a  $km \times km$  matrix  $Q$  of pseudodifferential operators in the class of the homogeneous zero order symbols  $I_h^0(X)$ , named the Calderón's projector over the modified Cauchy data of the  $\mathcal{C}^{\infty}$  functions belonging to the kernel of  $L$  (cf. [3], [9]). The principal symbol  $q$  of  $Q$  only depends on  $\sigma_0(L)$  ([3]) and is a  $km \times km$  matrix, which rank is supposed  $r$  (constant). This is always true for  $n \geq 3$ .

An elliptic boundary condition for the operator  $L$  is meant a  $r \times km$  matrix  $B$  of pseudodifferential operators in the class  $I_h^0(X)$  such that

$$B : \underbrace{\mathcal{C}^{\infty}(X, E) \otimes \cdots \otimes \mathcal{C}^{\infty}(X, E)}_{m\text{-times}} \longrightarrow \mathcal{C}^{\infty}(\tilde{F}),$$

where  $\tilde{F}$  is a  $r$ -dimensional vectorial sub-bundle of  $\underbrace{F \otimes \cdots \otimes F}_{m\text{-times}}$ , and the matrix  $bq$  has constant rank equal to  $r$  ([3]). The principal symbol  $b$  of  $B$  is a  $r \times km$  matrix. For those  $L$  and  $B$  it is said that the boundary problem  $L_B(L, B)$  is elliptic. Actually,  $L_B$  is the closed unbounded operator on  $L^2(M)$ , obtained as the closure of  $L$  acting on  $\mathcal{C}^{\infty}$  sections of  $E$  satisfying the boundary condition  $B$  on  $X$  ([3], [9]). By  $L_B^{-1}$  we mean the bounded operator which is the inverse of  $L_B$ , when it exists, and in this case we say that the problem  $L_B$  is invertible.



Let us denote with  $T$  the linear application which gives the Cauchy data

$$T : \mathcal{C}^\infty(M, E) \longrightarrow \underbrace{\mathcal{C}^\infty(X, E) \otimes \mathcal{C}^\infty(X, E) \otimes \cdots \otimes \mathcal{C}^\infty(X, E)}_{m\text{-times}}$$

$$u(x) = u(x', x_n) \mapsto Tu(x) = (u(x'), \partial_\nu u(x'), \dots, \partial_\nu^{m-1} u(x')),$$

where  $\nu$  is unitary outward normal vector to the boundary  $X$ . For each point  $x$  in a local chart of  $M$  intersecting  $X$ , it is written  $x = (x', x_n) \in M$  with  $x' \in X$  and  $x_n$  the  $X$ -normal coordinate.

The unique function

$$G(x, y) : M \times M \longrightarrow \text{Hom}(F, E)$$

which is linear from the  $F$ -fiber over  $y$  to the  $E$ -fiber over  $x$  satisfying

- (i)  $L(G(x, y)) = \delta(x, y)$ , being  $\delta(x, y)$  the Dirac delta function, and
- (ii)  $T(G(x, y)) \in \text{Ker}(B)$ , i. e. the Cauchy data of  $G(x, y)$  belong, as function of  $x$ , to the kernel of the boundary operator  $B$ ,

is called the Green function for the boundary problem  $L_B = (L, B)$ . This function  $G(x, y)$  is the kernel of the inverse operator  $L_B^{-1}$ . In what follows,  $G(x, y)$  will be written  $L_B^{-1}(x, y)$  when no confusion arises.

## 2.2 Trace class operators and Fredholm determinant

A compact operator  $A$  defined on a Hilbert space  $H$  is called to be a trace class operator if

$$\text{Tr}(|A|) = \sum_{j=1}^{\infty} \mu_j(A) < \infty, \quad (7)$$

where  $\mu_j(A)$ , the singular values of  $A$ , are the eigenvalues of  $|A| = \sqrt{A^*A}$ . The set of the trace class operators on  $H$  form an ideal denoted  $\mathcal{J}_1$ . If  $I$  denotes the identity operator on  $H$ , the Fredholm determinant of  $L = I - A$  was defined by (1) as

$$\det_1 L = \prod_{j=1}^{\infty} (1 - \lambda_j),$$

where  $\{\lambda_j(A)\}_j$  denotes the proper values of  $A$  when  $A$  is a trace class operator. Of course, its trace is given by

$$Tr(A) = \sum_{j=1}^{\infty} \lambda_j(A) < \infty.$$

The expression (7) defines a norm on  $\mathcal{J}_1$ , called the trace norm and denoted  $\|A(z)\|_1 = Tr(|A|)$ .

Also, the integral representation of the Fredholm determinant given in [8] will be used

$$det_1(I - A) = \exp \left\{ - \int_{\gamma} Tr [A(1 - zA)^{-1}] dz \right\}, \quad (8)$$

with  $\gamma : [0, 1] \longrightarrow \mathbb{C}$  a continuous path such that  $\gamma(0) = 0$ ,  $\gamma(1) = 1$  and that the operator  $(1 - zA)^{-1}$  exists and is bounded for all  $z$  in  $\gamma$ .

### Differentiability properties of the Fredholm determinant

In this paragraph some properties connected with the differentiability of the Fredholm determinants are recalled. The corresponding proofs can be found in [1].

#### Lemma 2.1.

Let  $A(z) : G \longrightarrow \mathcal{J}_1$  a holomorphic application over an open subset  $G$  of  $\mathbb{C}$  taking values on the ideal  $\mathcal{J}_1$  of the trace class operators equipped with the norm of  $\mathcal{L}(H)$ . Let us suppose that the trace norm  $\|A(z)\|_1$  of  $A(z)$  is bounded over each compact subset of  $G$ .

Then, the function  $det_1(I - A(z)) : G \longrightarrow \mathbb{C}$  is holomorphic.

#### Lemma 2.2.

Under the hypotheses of lemma 2.1 we have

- the derivative of  $A(z)$  is a trace class operator for all  $z \in G$ ;
- the function  $Tr(A(z))$  is holomorphic on  $G$ , and
- $\frac{d}{dz} [Tr(A(z))] = Tr \left[ \frac{d}{dz} A(z) \right]$ .

**Remark 2.3.** Since  $\mathcal{J}_1$  is not a closed subspace of  $\mathcal{L}(H)$  in the norm of the bounded operators, the first statement is not evident at all.

**Lemma 2.4.**

*Under the hypotheses of lemma 2.1 it results*

$$\frac{d}{dz} \ln(\det_1(I - A(z))) = -\text{Tr} \left[ (I - A(z))^{-1} \frac{d}{dz}(A(z)) \right].$$

**Remark 2.5.** *Let us notice the very close connection between this last lemma and the formula (8) given in [8].*

### 2.3 $\zeta$ -determinant

Let  $L$  be an endomorphism on a vectorial space of finite dimension. If  $\lambda_1, \lambda_2, \dots, \lambda_k$  are the eigenvalues of  $L$  repeated the times indicated by their multiplicities, the determinant of  $L$  is defined by  $\det_1 L \det L = \prod_{j=1}^k \lambda_j$ .

So much,

$$\ln \det L = \sum_{j=1}^k \ln \lambda_j = \frac{d}{ds} \left[ - \sum_{j=1}^k \lambda_j^{-s} \right] \Big|_{s=0} = - \frac{d}{ds} [\text{Tr}(L^{-s})] \Big|_{s=0},$$

for a suitable determination of the logarithm. From here it results

$$\det L = \exp \left\{ - \frac{d}{ds} \Big|_{s=0} [\text{Tr}(L^{-s})] \right\}. \quad (9)$$

Let us treat the case of a classical elliptic pseudodifferential operator  $L$  of order  $m > 0$  defined over the Hilbert space  $L^2(M)$ , if  $M$  is a compact manifold without boundary; or an elliptic differential operator with elliptic boundary conditions also defined over  $L^2(M)$ , when  $M$  is a compact differential manifold with boundary. Since  $L$  is an unbounded operator, it is clear that the product of the eigenvalues is divergent. In order to establish for this case a similar expression to (9) that allows to obtain a finite quantity as a function of these eigenvalues, it is necessary to define the generalized Riemann  $\zeta$ -function, associated to the operator  $L$ . To that end it is imperative to precise the notion of complex powers of  $L$ . Given a complex number  $s$ , one of the ways to define the operator  $L^{-s}$  is ([12], [13] and [15])

$$\left. \begin{aligned} L^{-s} &= \frac{i}{2\pi} \int_{\Gamma} \lambda^{-s} (L - \lambda I)^{-1} d\lambda, & \text{if } \text{Re}(s) > 0 \\ L^{-s} &= L^k \cdot L^{-(k+s)}, & \text{if } -k < \text{Re}(s) \leq -(k-1) \leq 0, \end{aligned} \right\} \quad (10)$$

with  $k \geq 1$  an integer number and  $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$  the path on the complex plane, where for some angle  $\theta$  each path is defined by

$$\begin{aligned}\Gamma_1 &= \{te^{i\theta}\}, \text{ varying } t \text{ from } \infty \text{ to } \epsilon > 0, \\ \Gamma_2 &= \{|\lambda| = \epsilon\} \text{ clockwise oriented, and} \\ \Gamma_3 &= \{te^{i\theta}\}, \text{ varying } t \text{ from } \epsilon \text{ to } \infty,\end{aligned}\tag{11}$$

assuming that there exists a cone of directions around the ray  $\arg \lambda = \theta$  in such a way that no eigenvalue of  $L$  belongs to the cone. In [12], [13] and [14] it was proved that the function  $Tr(L^{-s})$  is holomorphic in a half-plane and that admits a meromorphic extension to the whole complex  $s$ -plane, being analytic at  $s = 0$ . Then, the generalized Riemann  $\zeta$ -function, associated to  $L$  is defined by

$$\zeta(L, s) = Tr(L^{-s}).$$

Note its similitude with the numerical Riemann  $\zeta$ -function. In this way formula (9) gives the definition of the regularized determinant of the operator  $L$  by means of the generalized Riemann  $\zeta$ -function and that, in what follows, it will be denoted  $Det_\zeta L$ . Therefore,

$$Det_\zeta L = \exp \left\{ -\frac{d}{ds} \Big|_{s=0} \zeta(L, s) \right\}.\tag{12}$$

### 3 Proofs

#### Proofs of theorems 1.1 and 1.2

Next, both proofs are jointly exhibited since they have the same structure. The reader is advised about the necessity of keeping in mind the meaning of the notation  $L_t$  in each theorems.

Under the hypotheses the complex powers of  $L_t$  are given by ([12], [13] and [15])

$$\begin{aligned}L_t^{-s} &= \frac{i}{2\pi} \int_{\Gamma} \lambda^{-s} (L_t - \lambda)^{-1} d\lambda, & \text{if } \operatorname{Re}(s) > 0 \\ L_t^{-s} &= L_t^k \cdot L_t^{-(k+s)}, & \text{if } -k < \operatorname{Re}(s) \leq -(k-1) \leq 0,\end{aligned}$$

where  $k \geq 1$  is an integer and  $\Gamma$  is the curve described in (10).

Let  $k > \frac{n}{m}$  be an integer and  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) \geq k$ . According to [12], [13], [14], [15] and [17],  $L_t^{-s}$  is a trace class operator and its kernel is continuous on the diagonal of

$M$ . Since the complex powers depend analytically on the parameter  $s$  ([10], [12]), from lemma 2.2 it follows for  $\text{Re}(s) > k$  that

$$\begin{aligned}
\frac{d}{dt} \text{Tr}(L_t^{-s}) &= \frac{d}{dt} \text{Tr}[L_t^{k-s} L_t^{-k}] \\
&= \frac{d}{dt} \text{Tr} \left[ \frac{i}{2\pi} \int_{\Gamma} \lambda^{k-s} (L_t - \lambda)^{-1} L_t^{-k} d\lambda \right] \\
&= \text{Tr} \left\{ \frac{i}{2\pi} \int_{\Gamma} \lambda^{k-s} \left[ -(L_t - \lambda)^{-1} \frac{d}{dt}(L_t) (L_t - \lambda)^{-1} L_t^{-k} + \right. \right. \\
&\quad \left. \left. + (L_t - \lambda)^{-1} \left( \sum_{j=1}^k L_t^{-j+1} \frac{d}{dt}(L_t^{-1}) L_t^{-k+j} \right) \right] d\lambda \right\} \\
&= -\frac{i}{2\pi} \int_{\Gamma} \lambda^{k-s} \text{Tr} \left[ (L_t - \lambda)^{-1} \frac{d}{dt}(L_t) (L_t - \lambda)^{-1} L_t^{-k} d\lambda \right] + \\
&\quad + \frac{i}{2\pi} \sum_{j=1}^k \int_{\Gamma} \lambda^{k-s} \text{Tr} \left[ -(L_t - \lambda)^{-1} L_t^{-j+1} L_t^{-1} \frac{d}{dt}(L_t) L_t^{-1} L_t^{-k+j} \right] d\lambda.
\end{aligned}$$

By the cyclic property of the trace it can be written as

$$\begin{aligned}
\frac{d}{dt} \text{Tr}(L_t^{-s}) &= -\frac{i}{2\pi} \int_{\Gamma} \lambda^{k-s} \text{Tr} \left( (L_t - \lambda)^{-2} \frac{d}{dt}(L_t) L_t^{-k} \right) d\lambda - \\
&\quad - \frac{i}{2\pi} \sum_{j=1}^k \int_{\Gamma} \lambda^{k-s} \text{Tr} \left[ (L_t - \lambda)^{-1} \frac{d}{dt}(L_t) L_t^{-k-1} \right] d\lambda \\
&= \text{Tr} \left[ -\frac{i}{2\pi} \int_{\Gamma} \lambda^{k-s} (L_t - \lambda)^{-2} d\lambda \frac{d}{dt}(L_t) L_t^{-k} \right] - k \text{Tr} \left[ L_t^{k-s} \frac{d}{dt}(L_t) L_t^{-k-1} \right].
\end{aligned}$$

Integrating by parts and taking into account that  $\text{Re}(s) > k$ , we have

$$\begin{aligned}
\frac{d}{dt} \text{Tr}(L_t^{-s}) &= \text{Tr} \left[ (k-s) \frac{d}{dt}(L_t) L_t^{-s-1} - k \frac{d}{dt}(L_t) L_t^{-s-1} \right] \\
&= \text{Tr} \left[ -s \frac{d}{dt}(L_t) L_t^{-s-1} \right] \\
&= (-s) \cdot \text{Tr} \left[ \frac{d}{dt}(L_t) \cdot L_t^{-s-1} \right]. \tag{13}
\end{aligned}$$

As a function of  $s$  the r.h.s. of (13) has a meromorphic extension to the whole complex plane ([12], [13], [14], [15] and [17]) with only simple poles possibly localized at  $s = n - \frac{j}{m}$ , for  $j = 1, 2, \dots$ . In particular, at  $s = 0$  such extension is analytical.

Eventually, in virtue of definition of  $\zeta$ -determinant given by formula (12) and expression (13), it is clear that

$$\begin{aligned}\frac{d}{dt} \ln Det_{\zeta} L_t &= \frac{d}{dt} \left\{ -\frac{d}{ds} \Big|_{s=0} Tr(L_t^{-s}) \right\} \\ &= -\frac{d}{ds} \Big|_{s=0} \left\{ \frac{d}{dt} Tr(L_t^{-s}) \right\} \\ &= \frac{d}{ds} \Big|_{s=0} \left\{ s.Tr \left[ \frac{d}{dt}(L_t).L_t^{-s-1} \right] \right\}.\end{aligned}\tag{14}$$

□

#### Proof of corollary 1.4

In fact, from the integral representation (8) of the Fredholm determinant  $det_1$  it results that

$$\frac{d}{ds} \Big|_{s=0} \left\{ s.Tr \left[ \frac{d}{dt}(L_t).L_t^{-s-1} \right] \right\} = Tr \left[ \frac{d}{dt}(L_t).L_t^{-1} \right] = \frac{d}{dt} \ln det_1 (L_t.L_0^{-1}).$$

The conclusion follows straightforward after integrating the last equality from 0 to  $t_o$  and taking the exponential function. □

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