**DEPARTAMENTO DE MATEMÁTICA
DOCUMENTO DE TRABAJO**

“On Summability of Bilinear Operators”

Daniel Carando y Verónica Dimant

D.T.: N° 25

Noviembre 2001

On summability of bilinear operators

DANIEL CARANDO AND VERÓNICA DIMANT

Abstract

We study some properties of strongly and absolutely p -summing bilinear operators. We show that Hilbert-Schmidt bilinear mappings are both strongly and absolutely p -summing, for every $p \geq 1$. Giving an example of a strongly 1-summing bilinear mapping which fails to be weakly compact, we answer a question posed in [6]. We prove that, as in the linear case, every bilinear operator from $\mathcal{L}_\infty$ -spaces to an $\mathcal{L}_2$ -space is absolutely 2-summing.

1 Introduction

The great interest aroused by the theory of absolutely p -summing linear operators lead to a development of analogous concepts for multilinear mappings. Thus, absolute p -summability, p -domination and strong p -summability were introduced and studied by several authors (Pietsch, Alencar, Matos, Meléndez, Tonge, Schneider, Botelho, Dimant, among others).

The aim of this note is to explore more properties of these classes. All our results are stated for bilinear mappings, though many of them clearly remain true in the multilinear case.

The paper is organized as follows: in the second section it is shown that Hilbert-Schmidt bilinear operators are both strongly and absolutely p -summing for every $p \geq 1$. The third section deals with some examples that answer very natural questions about strong p -summability, among these an open problem from [6]. Section 4 is devoted to the relationship between different kinds of summability. Some situations are exhibited where every strongly p -summing bilinear mapping is absolutely p -summing. It is also proved that bilinear operators from $\mathcal{L}_\infty$ -spaces to $\mathcal{L}_2$ -spaces are absolutely 2-summing, as in the linear case.

Throughout E, E_1, E_2 and F will be Banach spaces over the real numbers. We will denote by $\mathcal{L}(E_1, E_2; F)$ the space of continuous bilinear operators from $E_1 \times E_2$ into F . The complete projective tensor product of E_1 and E_2 will be denoted by $E_1 \widehat{\otimes}_\pi E_2$. The closed unit ball of E will be noted B_E .

Recall that a linear operator $T \in \mathcal{L}(E; F)$ is **absolutely p -summing** if there exists a constant $K \geq 0$ such that for every $x_1, \dots, x_n \in E$, we have:

$$\left(\sum_{j=1}^n \|T(x_j)\|^p \right)^{\frac{1}{p}} \leq K \left(\sup_{\gamma \in B_{E'}} \sum_{j=1}^n |\gamma(x_j)|^p \right)^{\frac{1}{p}}$$

We can think that the supremum in the right hand side is either taken over elements of the dual space of E or over operators of the same kind of T (that is linear, in this case)

but with range the scalar field. These different ways of thinking give rise to the following definitions:

Definition 1.1 [1, 9] *A bilinear operator $T \in \mathcal{L}(E_1, E_2; F)$ is **absolutely p -summing** if there exists a constant $K \geq 0$ such that $x_1, \dots, x_n \in E_1$, $y_1, \dots, y_n \in E_2$, we have:*

$$\left(\sum_{j=1}^n \|T(x_j, y_j)\|^p \right)^{\frac{1}{p}} \leq K \left(\sup_{\gamma \in B_{E'_1}} \sum_{j=1}^n |\gamma(x_j)|^p \right)^{\frac{1}{p}} \left(\sup_{\eta \in B_{E'_2}} \sum_{j=1}^n |\eta(y_j)|^p \right)^{\frac{1}{p}}$$

We will denote by $\mathcal{L}_{as}^p(E_1, E_2; F)$ the space of absolutely p -summing bilinear operators, which is a Banach space if we consider the norm to be the infimum of all K that verify the inequality above.

Definition 1.2 [6] *A bilinear operator $T \in \mathcal{L}(E_1, E_2; F)$ is said to be **strongly p -summing** if there exists a constant $K \geq 0$ such that for every $x_1, \dots, x_n \in E_1$, $y_1, \dots, y_n \in E_2$, we have:*

$$\left(\sum_{j=1}^n \|T(x_j, y_j)\|^p \right)^{\frac{1}{p}} \leq K \sup_{\phi \in B_{\mathcal{L}(E, E_2)}} \left(\sum_{j=1}^n |\phi(x_j, y_j)|^p \right)^{\frac{1}{p}}$$

The space of all strongly p -summing bilinear operators will be denoted by $\mathcal{L}_{ss}^p(E_1, E_2; F)$ and the strongly p -summing norm is the infimum of all K verifying the inequality.

For general properties about absolutely p -summing linear operators and tensor products of Banach spaces we refer to [5] and [4].

We wish to thank Geraldo Botelho for helpful conversations and suggestions.

2 Summing properties of Hilbert-Schmidt bilinear operators

In this section, H_1 , H_2 and H will denote Hilbert spaces. A linear operator $T : H_1 \rightarrow H$ is said to be a Hilbert-Schmidt operator if for every (or for some) orthonormal basis $\{e_i\}_{i \in I}$ of H_1 we have

$$\sum_{i \in I} \|T(e_i)\|^2 < \infty.$$

In this case, the $H - S$ norm of T is $\|T\|_{H-S} = \left(\sum_{i \in I} \|T(e_i)\|^2 \right)^{1/2}$. It is known that for any p , absolutely p -summing operators between Hilbert spaces coincide with Hilbert-Schmidt operators.

Definition 2.1 *A bilinear mapping $T : H_1 \times H_2 \rightarrow H$ is Hilbert-Schmidt if $\sum_{i,j} \|T(e_i^1, e_j^2)\|^2 < \infty$ for every (or some) orthonormal basis $\{e_i^s\}_{i \in I_s}$ of H_s , $s = 1, 2$. The $H - S$ norm of T is $\|T\|_{H-S} = \left(\sum_{i,j} \|T(e_i^1, e_j^2)\|^2 \right)^{1/2}$.*

As in the linear case, the space of Hilbert-Schmidt bilinear operators $\mathcal{L}_{H-S}(E_1, E_2; F)$ is a Hilbert space with the $H-S$ norm.

Proposition 2.2 *If $T : H_1 \times H_2 \rightarrow H$ is a Hilbert-Schmidt bilinear operator, then T is strongly 1–summing (and therefore, p –summing for every p).*

Proof. If we consider the operator $T_0 : H \rightarrow \mathcal{L}_{H-S}(H_1, H_2; \mathbb{R})$ (between Hilbert spaces) given by $T_0(y)(x_1, x_2) = \langle T(x_1, x_2), y \rangle$, then T_0 results a Hilbert-Schmidt operator, since for an orthonormal basis $\{e_k\}_k$ of H we have

$$\begin{aligned} \sum_k \|T_0(e_k)\|_{H-S}^2 &= \sum_k \sum_{i,j} |T_0(e_k)(e_i^1, e_j^2)|^2 = \sum_{i,j} \sum_k |\langle T(e_i^1, e_j^2), e_k \rangle|^2 \\ &= \sum_{i,j} \|T(e_i^1, e_j^2)\|^2 = \|T\|_{H-S}^2. \end{aligned}$$

This means that the adjoint operator $T'_0 : \mathcal{L}_{H-S}(H_1, H_2; \mathbb{R}) \rightarrow H$ is also Hilbert-Schmidt and therefore absolutely 1–summing.

We now define the bilinear operator $\Lambda : H_1 \times H_2 \rightarrow \mathcal{L}_{H-S}(H_1, H_2; \mathbb{R})$ by $\Lambda(x, y) = \langle x, \cdot \rangle_{H_1} \langle y, \cdot \rangle_{H_2}$. If we see that $T = T'_0 \circ \Lambda$, since T'_0 is an absolutely 1–summing linear operator, T will turn out to be strongly 1–summing [6, Prop. 1.5 (iii)] (and strongly p –summing for all p [6, Prop. 1.4]). To see that $T = T'_0 \circ \Lambda$, note that for $x \in H_1$, $y \in H_2$ and $u \in H$,

$$\begin{aligned} \langle T'_0(\Lambda(x, y)), u \rangle_H &= \langle \Lambda(x, y), T_0(u) \rangle_{\mathcal{L}_{H-S}(H_1, H_2; \mathbb{R})} \\ &= \sum_{i,j} \Lambda(x, y)(e_i^1, e_j^2) \cdot T_0(u)(e_i^1, e_j^2) = \\ &= \sum_{i,j} x_i y_j \langle T(e_i^1, e_j^2), u \rangle = \langle T(x, y), u \rangle \end{aligned}$$

which completes the proof. ■

Unlike the linear case, the converse of the previous propositions does not hold: the bilinear form $\phi : \ell_2 \times \ell_2 \rightarrow \mathbb{R}$ given by $\phi(x, y) = \langle x, y \rangle$, is strongly p –summing for all p (as every scalar-valued bilinear form) but it is clearly not Hilbert-Schmidt.

For the analogous result for absolutely p –summing bilinear operators, note that the previous proposition and the fact that strongly 1–summing bilinear operators are absolutely 1–summing [6, Prop. 1.3 (ii)], assure that Hilbert-Schmidt bilinear operators are absolutely 1–summing. But we cannot conclude that they are absolutely p –summing for $p > 1$, since absolute 1–summability does not imply absolute p –summability (see the examples at the beginning of section 4). To obtain the desired result, we need the following lemma, which is a simplified version of [2, Lemma 3.2].

Lemma 2.3 *Let $T \in \mathcal{L}(E_1, E_2; F)$ and let $T_1 : E_1 \rightarrow \mathcal{L}(E_2; F)$ be the associated linear operator. If T_1 is absolutely p –summing, then so is T .*

Proposition 2.4 *If $T : H_1 \times H_2 \rightarrow H$ is a Hilbert-Schmidt bilinear operator, then T is absolutely p –summing for every p .*

Proof. Let $T_1 : H_1 \rightarrow \mathcal{L}(H_2; H)$ be the associated linear operator. For $x \in H_1$, $T_1(x)$ is a Hilbert-Schmidt operator, since

$$\begin{aligned} \sum_j \|T_1(x)(e_j^2)\|^2 &= \sum_j \|T(x, e_j^2)\|^2 = \sum_j \left\| \sum_i x_i T(e_i^1, e_j^2) \right\|^2 \\ &\leq \sum_j \left(\sum_i |x_i| \|T(e_i^1, e_j^2)\| \right)^2 \\ &\leq \sum_j \left(\sum_i |x_i|^2 \right) \left(\sum_i \|T(e_i^1, e_j^2)\|^2 \right) \\ &= \|x\|^2 \sum_j \sum_i \|T(e_i^1, e_j^2)\|^2 < +\infty. \end{aligned}$$

Since T_1 ranges inside $\mathcal{L}_{H-S}(H_2; H)$, we can consider its co-restriction $\hat{T}_1 : H_1 \rightarrow \mathcal{L}_{H-S}(H_2; H)$, which in fact is a Hilbert-Schmidt operator: $\sum_i \|\hat{T}_1(e_i^1)\|_{H-S}^2 = \sum_i \sum_j \|\hat{T}(e_i^1)(e_j^2)\|^2 = \sum_i \sum_j \|T(e_i^1, e_j^2)\|^2 < +\infty$. But Hilbert-Schmidt linear operators are absolutely p -summing for all p . This and the fact that $\mathcal{L}_{H-S}(H_2; H) \hookrightarrow \mathcal{L}(H_2; H)$ is continuous, imply that T_1 is absolutely p -summing for all p . Now the result is a consequence of Lemma 2.3. ■

A scalar version of the above result was proved by Matos in [10].

3 A multipurpose example

The bilinear operator

$$\begin{aligned} T : \ell_2 \times \ell_2 &\rightarrow \ell_2 \\ (x, y) &\mapsto (x_n y_n)_n \end{aligned}$$

factors through the inclusion $\ell_1 \hookrightarrow \ell_2$ and consequently is strongly p -summing for every p . However, we will see that the same formula produces operators which fail to be strongly p -summing, depending on the involved spaces. Moreover, this formula will allow us to solve the open problem posed in [6].

Thus, throughout this section the bilinear operator T will be given by $T(x, y) = (x_n y_n)_n$. First we consider T taking its values in ℓ_1 :

Example 3.1 $T : \ell_2 \times \ell_2 \rightarrow \ell_1$ is not strongly p -summing for any p .

If it were, the associated linear operator $T_1 : \ell_2 \rightarrow \mathcal{L}(\ell_2; \ell_1)$ would verify that, for any $x \in \ell_2$, $T_1(x) : \ell_2 \rightarrow \ell_1$ is an absolutely p -summing operator (see the first step in the proof of [6, Th. 2.2]). Since p -summing linear operators from ℓ_2 to ℓ_1 are 1-summing (see, for example, [5, Cor. 3.16]), it is enough to find $x \in \ell_2$ such that $T_1(x)$ is not absolutely 1-summing.

Let $x = (\frac{1}{j})_j$ and let $D_n = \{-1, 1\}^n$. We can consider each $w = (w_1, \dots, w_n) \in D_n$ as an element of ℓ_2 , identifying w with the vector $(w, 0, 0, \dots)$. If $T_1(x)$ is absolutely 1-summing, there must be a constant $K > 0$ such that

$$\sum_{w \in D_n} \|T_1(x)(w)\|_{\ell_1} \leq K \sup_{\gamma \in B_{\ell_2'}} \sum_{w \in D_n} |\gamma(w)|.$$

The left-hand side of the inequality is $\sum_{w \in D_n} \|T_1(x)(w)\|_{\ell_1} = \sum_{w \in D_n} (1 + \frac{1}{2} + \dots + \frac{1}{n}) = 2^n(1 + \frac{1}{2} + \dots + \frac{1}{n})$.

On the other hand, let $\varepsilon_j(w) = w_j$, $j = 1, \dots, n$ and μ_n the measure on D_n given by $\mu_n(\{w\}) = \frac{1}{2^n}$. By Khintchine's inequality [4, 8.5] we have

$$\begin{aligned} \sup_{\gamma \in B_{\ell_2'}} \sum_{w \in D_n} |\gamma(w)| &= \sup_{\gamma \in B_{\ell_2}} \sum_{w \in D_n} \left| \sum_{j=1}^n \varepsilon_j(w) \gamma_j \right| \\ &= \sup_{\gamma \in B_{\ell_2}} 2^n \int_{D_n} \left| \sum_{j=1}^n \varepsilon_j(w) \gamma_j \right| d\mu_n(w) \\ &\leq 2^n \sup_{\gamma \in B_{\ell_2}} \left(\sum_{j=1}^n |\gamma_j|^2 \right)^{\frac{1}{2}} = 2^n. \end{aligned}$$

Since there cannot be a constant such that $2^n(1 + \frac{1}{2} + \dots + \frac{1}{n}) \leq K 2^n$, we obtain the desired result.

We have seen that the previous bilinear mapping was not strongly p -summing for any p by finding x such that $T_1(x)$ was not absolutely p -summing for any p . A natural question arises: suppose that $T : E_1 \times E_2 \rightarrow F$ verifies that, for every $x \in E_1$ and $y \in E_2$, $T_1(x) : E_2 \rightarrow F$ and $T_2(y) : E_1 \rightarrow F$ are both absolutely p -summing operators. Does it mean that T is strongly p -summing? The following example shows that this is not the case:

Example 3.2 $T : \ell_1 \times \ell_2 \rightarrow \ell_1$ is not strongly 1-summing, although $T_1(x)$ and $T_2(y)$ are absolutely 1-summing for all $x \in \ell_1$ and $y \in \ell_2$.

For $y \in \ell_2$, $T_2(y) : \ell_1 \rightarrow \ell_1$ is absolutely 1-summing since it factors through ℓ_2 . Likewise, if $x \in \ell_1$, $T_1(x) : \ell_2 \rightarrow \ell_1$ is also absolutely 1-summing, $T_1(x)$ factoring through c_0 .

For $T : \ell_1 \times \ell_2 \rightarrow \ell_1$ to be strongly 1-summing it is necessary that the operators $T_n : \ell_1^n \times \ell_2^n \rightarrow \ell_1^n$ given by $T_n(x, y) = (x_j y_j)_{j=1}^n$ have their strongly 1-summing norms all bounded by a same constant K . To see that there is not such a constant we will use the double Khintchine's inequality [4, 34.8]:

$$a_p^{-2} \left(\sum_{i,j=1}^n |\alpha_{ij}|^2 \right)^{\frac{1}{2}} \leq \left(\int_{D_n} \int_{D_n} \left| \sum_{i,j=1}^n \alpha_{ij} \varepsilon_i(w) \varepsilon_j(w') \right|^p d\mu_n(w) d\mu_n(w') \right)^{\frac{1}{p}} \leq b_p^2 \left(\sum_{i,j=1}^n |\alpha_{ij}|^2 \right)^{\frac{1}{2}}$$

for $(\alpha_{ij})_{i,j} \subset \mathbb{R}$, $1 \leq p < \infty$, where $D_n = \{-1, 1\}^n$, $\varepsilon_i(w) = w_i$, $i = 1, \dots, n$ and $\mu_n(\{w\}) = \frac{1}{2^n}$ for $w \in D_n$. The constants a_p and b_p are those of the "simple" Khintchine's inequality.

Considering w and w' as elements of ℓ_1 and ℓ_2 respectively, we first note that

$$\sum_{\substack{w \in D_n \\ w' \in D_n}} \|T_n(w, w')\|_{\ell_1} = 2^{2n} n. \quad (1)$$

If we identify a bilinear form $\phi : \ell_1^n \times \ell_2^n \rightarrow \mathbb{R}$ with the matrix $(a_{ij})_{ij} = (\phi(e_i, e_j))_{ij}$ (so that $\phi(x, y) = \sum_{i,j=1}^n a_{ij}x_iy_j$) we have

$$\begin{aligned} \sup_{\|\phi\| \leq 1} \sum_{\substack{w \in D_n \\ w' \in D_n}} |\phi(w, w')| &= \sup_{\|\phi\| \leq 1} \sum_{\substack{w \in D_n \\ w' \in D_n}} \left| \sum_{i,j=1}^n a_{ij} \varepsilon_i(w) \varepsilon_j(w') \right| \\ &= \sup_{\|\phi\| \leq 1} 2^{2n} \int_{D_n} \int_{D_n} \left| \sum_{i,j=1}^n a_{ij} \varepsilon_i(w) \varepsilon_j(w') \right| d\mu_n(w) d\mu_n(w') \\ &\leq 2^{2n} \cdot \sup_{\|\phi\| \leq 1} \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Some calculations show that $\|\phi\|_{\mathcal{L}(\ell_1, \ell_2; \mathbb{R})} = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$ and therefore, if $\|\phi\|_{\mathcal{L}(\ell_1, \ell_2; \mathbb{R})} \leq 1$, then $\left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$ is at most $\sqrt{n}$ and

$$\sup_{\|\phi\| \leq 1} \sum_{\substack{w \in D_n \\ w' \in D_n}} |\phi(w, w')| \leq 2^{2n} \cdot \sqrt{n}$$

which cannot dominate expression (1).

With slight modifications it can be shown that the operator $T : \ell_1 \times \ell_p \rightarrow \ell_r$ given by $T(x, y) = (x_j y_j)_j$ is not strongly 1-summing for any $1 \leq r < p \leq 2$. Moreover, if T is well defined from $\ell_q \times \ell_s$ in ℓ_r , with $q \leq s$ and $r < p = \min\{2, s\}$, we can factor

$$\begin{array}{ccc} \ell_q \times \ell_s & \xrightarrow{T} & \ell_r \\ \uparrow \uparrow & \nearrow & \\ \ell_1 \times \ell_p & & \end{array}.$$

The diagonal arrow is not strongly 1-summing and so neither is T . In particular, we have that $T : \ell_1 \times \ell_p \rightarrow \ell_1$ is not strongly 1-summing for $1 < p < \infty$. The case $p = 1$ has particular interest, and it deserves a bit of introduction.

If $T : E_1 \times E_2 \rightarrow F$ is a bilinear operator such that its linearization $\tilde{T} : E_1 \widehat{\otimes}_\pi E_2 \rightarrow F$ is absolutely p -summing, then T is strongly p -summing [6]. In the same article, the author asks whether there exist a strongly p -summing bilinear operator whose linearization fails to be absolutely p -summing. Related to this question, she asks about the existence of a strongly p -summing bilinear operator which is not weakly compact. The following example answers both questions. A polynomial version of this example was proved by G. Botelho, who kindly communicated it to the authors.

Example 3.3 *The operator $T : \ell_1 \times \ell_1 \rightarrow \ell_1$ given by $T(x, y) = (x_j y_j)_j$ is strongly 1-summing but it is not weakly compact. Therefore, its linearization $\tilde{T} : \ell_1 \widehat{\otimes}_\pi \ell_1 \rightarrow \ell_1$ cannot be absolutely p -summing for any p .*

T is not weakly compact since $T(e_n, e_n) = e_n$ for $n \in \mathbb{N}$. To see that T is strongly 1–summing, recall that for $\phi \in \mathcal{L}(\ell_1, \ell_1; \mathbb{R})$, $\phi(x, y) = \sum_{i,j} a_{ij} x_i y_j$, we have that $\|\phi\| = \sup_{i,j} |a_{ij}|$. Therefore, we must show that there exists $K > 0$ such that

$$\sum_{k=1}^N \sum_{n=1}^{\infty} |x_n^k y_n^k| \leq K \sup_{|a_{ij}| \leq 1} \sum_{k=1}^N \left| \sum_{i,j} a_{ij} x_i^k y_j^k \right|$$

for $(x^k)_k, (y^k)_k \subset \ell_1$. Clearly, it is enough to show this inequality for m instead of ∞ , as long as the same K works for all m . We have

$$\sum_{k=1}^N \sum_{n=1}^m |x_n^k y_n^k| \leq \sum_{k=1}^N \left(\sum_{n=1}^m |x_n^k|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^m |y_n^k|^2 \right)^{\frac{1}{2}}$$

and by Khintchine's inequality (used twice), we get

$$\begin{aligned} \sum_{k=1}^N \sum_{n=1}^m |x_n^k y_n^k| &\leq 2 \sum_{k=1}^N \left(\int_{D_m} \left| \sum_{i=1}^m x_i^k \varepsilon_i(w) \right| d\mu_m(w) \int_{D_m} \left| \sum_{j=1}^m y_j^k \varepsilon_j(w') \right| d\mu_m(w') \right) \\ &= 2 \sum_{k=1}^N \int_{D_m} \int_{D_m} \left| \sum_{i,j=1}^m x_i^k y_j^k \varepsilon_i(w) \varepsilon_j(w') \right| d\mu_m(w) d\mu_m(w') \\ &= 2 \int_{D_m} \int_{D_m} \sum_{k=1}^N \left| \sum_{i,j=1}^m x_i^k y_j^k \varepsilon_i(w) \varepsilon_j(w') \right| d\mu_m(w) d\mu_m(w') \\ &\leq 2 \sup_{\substack{w \in D_m \\ w' \in D_m}} \sum_{k=1}^N \left| \sum_{i,j=1}^m x_i^k y_j^k \varepsilon_i(w) \varepsilon_j(w') \right| \\ &\leq 2 \sup_{|a_{ij}| \leq 1} \sum_{k=1}^N \left| \sum_{i,j=1}^m a_{ij} x_i^k y_j^k \right|. \end{aligned}$$

Then T is strongly 1–summing.

4 Relations between different kinds of summability

In order to find relations between strong and absolute p –summability, we must look first at the scalar-valued case, as the following straightforward lemma states:

Lemma 4.1 *Every scalar-valued bilinear form on $E_1 \times E_2$ is absolutely p –summing if and only if $\mathcal{L}_{ss,p}(E_1, E_2; F) \subset \mathcal{L}_{as,p}(E_1, E_2; F)$ for every Banach space F .*

For $p = 1$ these equivalent conditions always hold [1, Th. 3.10], but for $p > 1$ the situation is different.

If $p \geq 2$, the bilinear form $\phi : \ell_2 \times \ell_2 \rightarrow \mathbb{R}$ given by $\phi(x, y) = \langle x, y \rangle$ is not absolutely p –summing, since $(e_n)_n$ is weakly p –summing but $\phi(e_n, e_n) = 1$ for all n .

For $1 < p < 2$, let q be such that $\frac{1}{p} + \frac{1}{q} = 1$ and consider the bilinear form $\phi : \ell_q \times \ell_q \rightarrow \mathbb{R}$ given by $\phi(x, y) = \sum_n a_n x_n y_n$, where $(a_n)_n \in \ell_{\frac{p}{2-p}} \setminus \ell_p$. $(e_n)_n$ weakly p -summing but $(\phi(e_n, e_n))_n = (a_n)_n$ which is not p -summing. Therefore, ϕ is not absolutely p -summing.

The previous examples also show that the absolutely p -summing bilinear operators do not share the inclusion property enjoyed by absolutely p -summing linear operators. Strongly p -summing operator satisfy this property [6].

It is known that every linear operator from an $\mathcal{L}_1$ -space to an $\mathcal{L}_2$ -space is absolutely 1-summing and every linear operator from an $\mathcal{L}_\infty$ -space to an $\mathcal{L}_p$ -space is absolutely 2-summing for $1 \leq p \leq 2$. These facts, Lemmata 2.3 and 4.1 give the following

Proposition 4.2 1) If E_1 is a $\mathcal{L}_1$ -space and E_2 is an $\mathcal{L}_2$ -space, then $\mathcal{L}(E_1, E_2; \mathbb{R}) = \mathcal{L}_{as,p}(E_1, E_2; \mathbb{R})$ for all $p \geq 1$. Therefore, $\mathcal{L}_{ss,p}(E_1, E_2; F) \subset \mathcal{L}_{as,p}(E_1, E_2; F)$ for every Banach space F and every $p \geq 1$.

2) If E_1 is a $\mathcal{L}_\infty$ -space and E_2 is an $\mathcal{L}_q$ -space for some $2 \leq q \leq \infty$, then $\mathcal{L}(E_1, E_2; \mathbb{R}) = \mathcal{L}_{as,p}(E_1, E_2; \mathbb{R})$ for all $p \geq 2$. Therefore, $\mathcal{L}_{ss,p}(E_1, E_2; F) \subset \mathcal{L}_{as,p}(E_1, E_2; F)$ for every Banach space F and every $p \geq 2$.

The second statement of the previous proposition can be improved for $q = \infty$. First, some definitions. A bilinear operator $T : E_1 \times E_2 \rightarrow F$ is **absolutely $(p; q)$ -summing** if there exists $K \geq 0$ such that for every $x_1, \dots, x_n \in E_1$, $y_1, \dots, y_n \in E_2$, we have:

$$\left(\sum_{j=1}^n \|T(x_j, y_j)\|^p \right)^{\frac{1}{p}} \leq K \left(\sup_{\gamma \in B_{E_1'}} \sum_{j=1}^n |\gamma(x_j)|^q \right)^{\frac{1}{q}} \left(\sup_{\eta \in B_{E_2'}} \sum_{j=1}^n |\eta(y_j)|^q \right)^{\frac{1}{q}}.$$

It is clear that an absolutely $(p_1; q_1)$ -summing operator is also absolutely $(p_2; q_2)$ -summing whenever $1 \leq p_1 \leq p_2$ and $1 \leq q_2 \leq q_1$. Absolutely $(\frac{p}{2}; p)$ -summing operators were called p -**dominated** in [11]. In the same article, it was proved that p -dominated operators satisfy the inclusion property: p_1 -domination implies p_2 -domination for $p_2 \geq p_1$. We also have

Proposition 4.3 If $T : E_1 \times E_2 \rightarrow F$ is 2-dominated, then T is absolutely p -summing for all p .

Proof. T being 2-dominated means that T is absolutely $(1; 2)$ -summing and then absolutely $(p; q)$ -summing for all $p \geq 1$ and $q \leq 2$. In particular, T is absolutely p -summing for $1 \leq p \leq 2$.

By the inclusion property, T is also 3-dominated ($\equiv$ absolutely $(\frac{3}{2}; 3)$ -summing) and then absolutely p -summing for $\frac{3}{2} \leq p \leq 3$. We can go on like this for ever and obtain the absolute p -summability of T for all p . ■

Recall that the w_2 -norm on the tensor product $E_1 \otimes E_2$ is defined as

$$w_2(s) = \inf \left\{ \left(\sup_{\gamma \in B_{E_1'}} \sum_{j=1}^n |\gamma(x_j)|^2 \right)^{\frac{1}{2}} \left(\sup_{\eta \in B_{E_2'}} \sum_{j=1}^n |\eta(y_j)|^2 \right)^{\frac{1}{2}} \right\},$$

where the infimum is taken over all representations $s = \sum_{j=1}^n x_j \otimes y_j$ [4, Def. 12.7]. Now we can state the tensor form of Grothendieck's inequality [4, 14.5]:

If E_1 is a $\mathcal{L}_{\infty, \lambda}$ -space and E_2 is a $\mathcal{L}_{\infty, \lambda'}$ -space, then $w_2 \leq \pi \leq K_G \lambda \lambda' w_2$ on $E_1 \otimes E_2$, where K_G is Grothendieck's constant.

Since w_2 -continuous bilinear forms are 2-dominated (actually, the reverse inclusion also holds [3, Th. 2.2]), Grothendieck's inequality and Proposition 4.3 give the following corollary, which was proved in [2] for $C(K)$ and in [11] for c_0 :

Corollary 4.4 *If E_1 and E_2 are $\mathcal{L}_{\infty}$ -spaces, every bilinear form $\phi : E_1 \times E_2 \rightarrow \mathbb{R}$ is 2-dominated (and absolutely p -summing for all p).*

Consequently, if E_1 and E_2 are $\mathcal{L}_{\infty}$ -spaces, for all Banach space F and all $p \geq 1$, $\mathcal{L}_{ss,p}(E_1, E_2; F) \subset \mathcal{L}_{as,p}(E_1, E_2; F)$.

As we mentioned before, every linear operator from an $\mathcal{L}_{\infty}$ -space to an $\mathcal{L}_2$ -space is absolutely 2-summing. We show that this also holds for bilinear operators. First, we prove the following:

Lemma 4.5 *Let $T : E_1 \times E_2 \rightarrow \ell_2^n$ be a continuous bilinear mapping, $T = (T_j)_{j=1, \dots, n}$. If $x^1, \dots, x^N \in E_1$ and $y^1, \dots, y^N \in E_2$,*

$$\int_{D_n} \left| \sum_{k=1}^N \sum_{j=1}^n \varepsilon_j(w) T_j(x^k, y^k) \right| d\mu_n(w) \leq \|T\| \left\| \sum_{k=1}^N x^k \otimes y^k \right\|_{E_1 \hat{\otimes}_{\pi} E_2}$$

Proof. Define $S : \ell_2^n \rightarrow \ell_1^{2n} = \ell_1(D_n)$ by $S(z) = \frac{1}{2^n} \left(\sum_{j=1}^n \varepsilon_j(w) z_j \right)_{w \in D_n}$. Since

$$\begin{aligned} \|S(z)\| &= \frac{1}{2^n} \sum_{w \in D_n} \left| \sum_{j=1}^n \varepsilon_j(w) z_j \right| = \int_{D_n} \left| \sum_{j=1}^n \varepsilon_j(w) z_j \right| d\mu_n(w) \\ &\leq \left(\int_{D_n} \left| \sum_{j=1}^n \varepsilon_j(w) z_j \right|^2 d\mu_n(w) \right)^{\frac{1}{2}} = \left(\sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}} = \|z\|, \end{aligned}$$

we have that $\|S\| \leq 1$ (in fact, $\|S\| = 1$). Therefore, the bilinear operator $S \circ T : E_1 \times E_2 \rightarrow \ell_1^{2n}$ verifies $\|S \circ T\| \leq \|T\|$ and is given by

$$(S \circ T)(x, y) = \frac{1}{2^n} \left(\sum_{j=1}^n \varepsilon_j(w) T_j(x, y) \right)_{w \in D_n}.$$

Consequently,

$$\begin{aligned} \int_{D_n} \left| \sum_{k=1}^N \sum_{j=1}^n \varepsilon_j(w) T_j(x^k, y^k) \right| d\mu_n(w) &= \frac{1}{2^n} \sum_{w \in D_n} \left| \sum_{k=1}^N \sum_{j=1}^n \varepsilon_j(w) T_j(x^k, y^k) \right| \\ &= \left\| \sum_{k=1}^N (S \circ T)(x^k, y^k) \right\|_{\ell_1^{2n}} \end{aligned}$$

$$\begin{aligned}
&= \left\| (S \circ T) \left(\sum_{k=1}^N x^k \otimes y^k \right) \right\|_{\ell_1^{2n}} \\
&\leq \|T\| \left\| \sum_{k=1}^N x^k \otimes y^k \right\|_{E_1 \widehat{\otimes}_\pi E_2}
\end{aligned}$$

as we wanted to prove. ■

Theorem 4.6 *Let E_1 be an $\mathcal{L}_{\infty, \lambda}$ -space, E_2 an $\mathcal{L}_{\infty, \lambda'}$ -space and F an $\mathcal{L}_{2, \mu}$ -space. Then, every bilinear operator $T : E_1 \times E_2 \rightarrow F$ is absolutely 2-summing and $\|T\|_{as, 2} \leq 2K_G \lambda \lambda' \mu \|T\|$ (where K_G is Grothendieck's constant).*

Proof. It is enough to show that every bilinear operator $T : E_1 \times E_2 \rightarrow \ell_2^n$ is absolutely 2-summing with $\|T\|_{as, 2} \leq 2K_G \lambda \lambda' \|T\|$. We take $x^1, \dots, x^N \in E_1$, $y^1, \dots, y^N \in E_2$. If we denote $T = (T_j)_{j=1, \dots, n}$, we have

$$\left(\sum_{k=1}^N \left\| T(x^k, y^k) \right\|_{\ell_2^n}^2 \right)^{\frac{1}{2}} = \left(\sum_{k=1}^N \sum_{j=1}^n \left(T_j(x^k, y^k) \right)^2 \right)^{\frac{1}{2}}$$

and by double Khintchine's inequality,

$$\begin{aligned}
\left(\sum_{k=1}^N \left\| T(x^k, y^k) \right\|_{\ell_2^n}^2 \right)^{\frac{1}{2}} &\leq a_1^2 \int_{D_N} \int_{D_n} \left| \sum_{k=1}^N \sum_{j=1}^n T_j(x^k, y^k) \varepsilon_j(w) \varepsilon_k(w') \right| d\mu_n(w) d\mu_N(w') \\
&= 2 \int_{D_N} \int_{D_n} \left| \sum_{k=1}^N \sum_{j=1}^n \varepsilon_j(w) T_j(\varepsilon_k(w') x^k, y^k) \right| d\mu_n(w) d\mu_N(w') \\
&\leq 2 \int_{D_N} \|T\| \left\| \sum_{k=1}^N \varepsilon_k(w') x^k \otimes y^k \right\|_{E_1 \widehat{\otimes}_\pi E_2} d\mu_N(w') \quad (2)
\end{aligned}$$

The last inequality is due to the previous lemma. By Grothendieck's inequality,

$$\begin{aligned}
\left\| \sum_{k=1}^N \varepsilon_k(w') x^k \otimes y^k \right\|_{E_1 \widehat{\otimes}_\pi E_2} &\leq K_G \lambda \lambda' w_2 \left(\sum_{k=1}^N \varepsilon_k(w') x^k \otimes y^k \right) \\
&\leq K_G \lambda \lambda' \left(\sup_{\gamma \in B_{E'_1}} \sum_{j=1}^n |\gamma(x_j)|^2 \right)^{\frac{1}{2}} \left(\sup_{\eta \in B_{E'_2}} \sum_{j=1}^n |\eta(y_j)|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

and this completes the proof. ■

Observe that in the proof, the fact that E_1 and E_2 are $\mathcal{L}_\infty$ -spaces is only necessary for the last step. That is, we can reach expression (2) without any assumption on the spaces E_1 and E_2 . Since

$$\begin{aligned}
\left\| \sum_{k=1}^N \varepsilon_k(w') x^k \otimes y^k \right\|_{E_1 \widehat{\otimes}_\pi E_2} &\leq \sup_{\phi \in B_{\mathcal{L}(E_1, E_2)}} \sum_{k=1}^N \left| \phi(\varepsilon_k(w') x^k, y^k) \right| \\
&= \sup_{\phi \in B_{\mathcal{L}(E_1, E_2)}} \sum_{k=1}^N \left| \phi(x^k, y^k) \right|,
\end{aligned}$$

we obtain the following theorem, where “strongly $(2; 1)$ –summing” is defined in the obvious way.

Theorem 4.7 *If F an $\mathcal{L}_{2,\mu}$ –space, then every bilinear operator $T : E_1 \times E_2 \rightarrow F$ is strongly $(2; 1)$ –summing and $\|T\|_{ss,(2;1)} \leq 2K_{G\mu} \|T\|$.*

References

- [1] R. ALENCAR and M. MATOS, Some classes of multilinear mappings between Banach spaces. Pub. Dep. An. Mat. Univ. Complut. Madrid **12** (1989)
- [2] G. BOTELHO, Cotype and absolutely summing multilinear mappings and homogeneous polynomials. Proc. Royal Irish Acad **97A** (2) (1997) 145-153
- [3] J. CASTILLO, R. GARCÍA and J. A. JARAMILLO, Extension of bilinear forms on Banach spaces. Proc. Amer. Math. Soc. **129** (2001) 3647-3656
- [4] A. DEFANT and K. FLORET, Tensor norms and operator ideals. North-Holland Mathematics Studies 1993
- [5] J. DIESTEL, H. JARCHOW and A. TONGE, Absolutely summing operators. Cambridge University Press 1995
- [6] V. DIMANT, Strongly p –summing multilinear operators. Preprint
- [7] S. DINEEN, Complex analysis on infinite dimensional spaces. Springer-Verlag, London 1999
- [8] J. LINDENSTRAUSS and L. TZAFRIRI, Classical Banach Spaces I. Springer 1977
- [9] M. MATOS, Absolutely summing holomorphic mappings. An. Acad. Bras. Ci. **68** (1996) 1-13
- [10] M. MATOS, On a Pietsch’s question about Hilbert-Schmidt multilinear mappings. Preprint.
- [11] Y. MELENDEZ and A. TONGE, Polynomials and the Pietsch domination theorem. Math. Proc. Royal Irish Acad. **99A** (2) (1999) 195-212
- [12] A. PIETSCH, Ideals of multilinear functionals (designs of a theory). Proc. of the 2nd Int. Conf. on Operator Algebras, Ideals and their applications in Theoretical Physics (1983) 185-199
- [13] B. SCHNEIDER, On absolutely p –summing and related multilinear mappings. Wissenschaftliche Zeitschrift der Brandenburger Landeshochschule **35** (1991) 105-117.

Departamento de Matemática y Ciencias,
 Universidad de San Andrés,
 Vito Dumas 284 (1644)
 Victoria, Argentina.
 daniel@udesa.edu.ar, vero@udesa.edu.ar