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STRONGLY $P-$SUMMING MULTILINEAR OPERATORS

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Abstract

We define the concept of strongly $p-$summing multilinear operators and prove that they have properly defined Aron-Berner extensions. Some properties and examples are shown.

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Introduction

The theory of absolutely $p-$summing linear operators is a deeply studied field of mathematical analysis. Many authors have developed related concepts for multilinear operators and homogeneous polynomials. Briefly, there have been two definitions in the multilinear case (as well as in the polynomial case): absolutely $p-$summing $m-$linear operators ([1, 8]) and $r-$dominated $m-$linear operators ([9]).

In the first case, it has been defined a $p-$summing norm that makes the space of absolutely $p-$summing $m-$linear operators a Banach space. Some other properties of absolutely $p-$summing linear operators remain true in the multilinear case. However, there is no analogous of a Pietsch domination theorem.

On the other hand, the concept of $r-$dominated $m-$linear operator has been defined in order to verify a Pietsch domination theorem, but the resulting space is not always a Banach space (if $r < m$ there is a only a quasinorm instead of a norm).

Here, we introduce a new attempt of lifting properties of absolutely $p-$summing linear operators to the multilinear and polynomial cases. The space of strongly $p-$summing $m-$linear operators is a Banach space and this kind of multilinear operators verify some analogous properties of those of the linear case (for instance, there is a sort of Pietsch domination theorem). Anyway, our motivation of this new definition is not just that. Our aim is trying to translate to the multilinear case the fact that absolutely $p-$summing linear operators are weakly compact. Recall that a linear operator $T \in \mathcal{L}(E, F)$ is weakly compact if and only if the bitranspose T'' is $F-$valued. For multilinear operators the analogous concepts are not equivalent. If an $m-$linear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is weakly compact then its Aron-Berner extension $\overline{T}$ is $F-$valued, but the reciprocal is not true. Following [6], this last property is more likely to be translated to the multilinear case than weak compactness. Thus, our motivation at defining strongly $p-$summing $m-$linear operators is that these operators verify that their Aron-Berner extension is $F-$valued.

Throughout $E, E_1, ..., E_m, F$ will be Banach spaces over the real numbers. We will denote by $\mathcal{L}(E_1, ..., E_m; F)$ the space of m −linear operators from $E_1 \times \cdots \times E_m$ into F. In the case $E_1 = \cdots = E_m = E$, we will simply write $\mathcal{L}(^m E; F)$ and $\mathcal{L}(E_1, ..., E_m)$ will mean $\mathcal{L}(E_1, ..., E_m; \mathbb{R})$. Similarly, the space of continuous $m-$homogeneous polynomials from E into F will be denoted by $\mathcal{P}(^m E; F)$ (with $\mathcal{P}(^m E) = \mathcal{P}(^m E; \mathbb{R})$) and the complete projective tensor product of $E_1, ..., E_m$ will be denoted by $E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_m$. The closed unit ball of E will be noted B_E. Since we will compare at some points our new concepts with the previously realized, let us recall the definition of Alencar and Matos (see [1], also [8]) of absolutely $p-$summing multilinear operators. An $m-$linear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is **absolutely $p-$summing** if for any weakly $p-$summable sequences $(x_n^i)_{n\in\mathbb{N}} \subset E_i$, $i = 1, ..., m$, the sequence $\left(T(x_n^1, ..., x_n^m)\right)_{n\in\mathbb{N}}$ is strongly $p-$summable in F. Equivalently, there exists a constant $K \geq 0$ such that for every $x_1^i, ..., x_n^i \in E_i$, $(i = 1, ..., m)$ we have:

$$\left(\sum_{j=1}^{n} \left\|T\left(x_j^1, ..., x_j^m\right)\right\|^p\right)^{\frac{1}{p}} \leq K \left(\sup_{\gamma_1 \in B_{E_1'}} \sum_{j=1}^{n} \left|\gamma_1\left(x_j^1\right)\right|^p\right)^{\frac{1}{p}} \cdots \left(\sup_{\gamma_m \in B_{E_m'}} \sum_{j=1}^{n} \left|\gamma_m\left(x_j^m\right)\right|^p\right)^{\frac{1}{p}}$$

We will denote by $\mathcal{L}_{as}^p(E_1, ..., E_m; F)$ the space of absolutely $p-$summing $m-$linear operators, which is a Banach space if we consider as the norm the infimum of all K that verify the inequality above.

The paper is organized as follows. In Section 1 we introduce the definition of strongly $p-$summing $m-$linear operators, some examples and properties (much of them similar to the linear case) are shown. Also we pose an important open question. In Section 2 we state and prove the result about Aron-Berner extensions of strongly $p-$summing $m-$linear operators and in Section 3 we briefly discuss some properties of strongly $p-$summing $m-$homogeneous polynomials.

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1. General properties

In order to introduce our definition of strongly $p-$summing multilinear operators, let us take a look at one of the equivalent definitions in the linear case. A linear operator $T \in \mathcal{L}(E; F)$ is **absolutely $p-$summing** if there exists a constant $K \geq 0$ such that for every $x_1, ..., x_n \in E$, we have:

$$\left(\sum_{j=1}^{n} \|T(x_j)\|^p\right)^{\frac{1}{p}} \leq K \left(\sup_{\gamma \in B_{E'}} \sum_{j=1}^{n} |\gamma(x_j)|^p\right)^{\frac{1}{p}}$$

We can think that the supremum in the right hand side is taken over elements of the dual space of E (this is the way of thinking of the authors previously cited) or we can think that the supremum is taken over operators of the same kind of T (that is linear, in this case) but with range the scalar field. Developing this idea we arrive to the next definition.

Definition 1.1 *An m −linear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is said to be* **strongly p−summing** *if there exists a constant $K \geq 0$ such that for every $x_1^i, ..., x_n^i \in E_i$, $(i = 1, ..., m)$ we have:*

$$\left(\sum_{j=1}^{n} \left\|T\left(x_j^1, ..., x_j^m\right)\right\|^p\right)^{\frac{1}{p}} \leq K \sup_{\phi \in B_{\mathcal{L}(E_1,...,E_m)}} \left(\sum_{j=1}^{n} \left|\phi\left(x_j^1, ..., x_j^m\right)\right|^p\right)^{\frac{1}{p}}$$

As in the linear case we have a domination equivalence.

Proposition 1.2 *Let $T \in \mathcal{L}(E_1, ..., E_m; F)$. The following assertions are equivalent:*

(i) T is strongly p−summing.

(ii) There exist a regular probability measure μ on $\left(B_{\mathcal{L}(E_1,...,E_m)}, w^*\right)$ and a constant $K \geq 0$ such that for every $\left(x^1, ..., x^m\right) \in E_1 \times \cdots \times E_m$, we have

$$\left\|T\left(x^1, ..., x^m\right)\right\| \leq K \left(\int_{B_{\mathcal{L}(E_1,...,E_m)}} \left|\phi\left(x^1, ..., x^m\right)\right|^p \, d\mu(\phi)\right)^{\frac{1}{p}}$$

(iii) If the sequence $\left\{x_j^1 \otimes \cdots \otimes x_j^m\right\}_{j\in\mathbb{N}}$ is weakly p−summable in $E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_m$, then the sequence $\left\{T\left(x_j^1, ..., x_j^m\right)\right\}_{j\in\mathbb{N}}$ is strongly p−summable in F.

Proof. To simplify the notation, let us write $\mathcal{L}^m$ instead of $\mathcal{L}(E_1, ..., E_m)$.

$(i) \Rightarrow (ii)$ Consider the sets

$$\begin{aligned} F_1 &= \left\{f \in C(B_{\mathcal{L}^m}, w^*) : \sup_{\phi \in B_{\mathcal{L}^m}} f(\phi) < 1\right\} \\ F_2 &= co\left\{f \in C(B_{\mathcal{L}^m}, w^*) : f = \left|x^1 \otimes \cdots \otimes x^m\right|^p \text{ where } \left\|T(x^1, ..., x^m)\right\| = K\right\} \end{aligned}$$

It is clear that F_1 and F_2 are convex sets and F_1 is an open set. Let´s see that $F_1 \cap F_2 = \emptyset$. If $f \in F_2$ then $f = \sum_{j=1}^n \alpha_j \left|x_j^1 \otimes \cdots \otimes x_j^m\right|^p$ with $\sum_{j=1}^n \alpha_j = 1$, $0 \leq \alpha_j \leq 1$, $\left\|T(x_j^1, ..., x_j^m)\right\| = K$. In this case $\sup_{\phi \in B_{\mathcal{L}^m}} f(\phi) \geq 1$. Indeed,

$$\begin{aligned} \sup_{\phi \in B_{\mathcal{L}^m}} f(\phi) &= \sup_{\phi \in B_{\mathcal{L}^m}} \sum_{j=1}^{n} \alpha_j \left|\phi\left(x_j^1, ..., x_j^m\right)\right|^p = \sup_{\phi \in B_{\mathcal{L}^m}} \sum_{j=1}^{n} \left|\phi\left(\alpha_j^{1/p} x_j^1, ..., x_j^m\right)\right|^p \\ &\geq \frac{1}{K^p} \sum_{j=1}^{n} \left\|T\left(\alpha_j^{1/p} x_j^1, ..., x_j^m\right)\right\|^p = \frac{1}{K^p} \sum_{j=1}^{n} \alpha_j K^p = 1 \end{aligned}$$

By Hahn-Banach theorem, there exists a measure $\mu \in C(B_{\mathcal{L}^m}, w^*)'$, $\|\mu\| = 1$ such that

$$\int_{B_{\mathcal{L}^m}} f_1(\phi) \, d\mu(\phi) \leq \lambda \leq \int_{B_{\mathcal{L}^m}} f_2(\phi) \, d\mu(\phi)$$

for all $f_1 \in F_1$ and $f_2 \in F_2$. Since all non negative functions belong to F_1, μ should be positive. Since the unit ball of $C(B_{\mathcal{L}^m}, w^*)$ is contained in F_1, λ should be greater than

or equal to 1. Thus, for all $\left(x^1,...,x^m\right)\in E_1\times\cdots\times E_m$ with$\left\|T(x^1,...,x^m)\right\|=K$ we have that

$$\int_{B_{\mathcal{L}^m}}\left|\phi\left(x^1,...,x^m\right)\right|^p\ d\mu(\phi)\geq 1$$

so it follows that, for all $\left(x^1,...,x^m\right)\in E_1\times\cdots\times E_m$,

$$K^p\int_{B_{\mathcal{L}^m}}\left|\phi\left(x^1,...,x^m\right)\right|^p\ d\mu(\phi)\geq\left\|T(x^1,...,x^m)\right\|^p.$$

$(ii)\Rightarrow(i)$

$$\sum_{j=1}^{n}\left\|T\left(x_j^1,...,x_j^m\right)\right\|^p\leq K^p\int_{B_{\mathcal{L}^m}}\sum_{j=1}^{n}\left|\phi\left(x_j^1,...,x_j^m\right)\right|^p d\mu(\phi)\leq K^p\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{n}\left|\phi\left(x_j^1,...,x_j^m\right)\right|^p$$

$(i)\Rightarrow(iii)$ By the closed graph theorem, if $\left\{x_j^1\otimes\cdots\otimes x_j^m\right\}_{j\in\mathbb{N}}$ is weakly $p-$summable, then $\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{\infty}\left|\phi\left(x_j^1,...,x_j^m\right)\right|^p<\infty$. So, if $\left\{x_j^1\otimes\cdots\otimes x_j^m\right\}_{j\in\mathbb{N}}$ is weakly $p-$summable, we have, for all $M\in\mathbb{N}$,

$$\sum_{j=1}^{M}\left\|T\left(x_j^1,...,x_j^m\right)\right\|^p\leq K^p\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{M}\left|\phi\left(x_j^1,...,x_j^m\right)\right|^p\leq K^p\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{\infty}\left|\phi\left(x_j^1,...,x_j^m\right)\right|^p<\infty.$$

$(iii)\Rightarrow(i)$ Suppose that (i) is false. Then, for every $k>0$ there exists a finite sequence $\left\{x_j^1(k)\otimes\cdots\otimes x_j^m(k)\right\}_{j=1}^{n(k)}$ so that

$$\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{n(k)}\left|\phi\left(x_j^1(k),...,x_j^m(k)\right)\right|^p=1\text{ and }\sum_{j=1}^{n(k)}\left\|T\left(x_j^1(k),...,x_j^m(k)\right)\right\|^p\geq k.$$

For each $k=N^2$, obtain the corresponding finite sequence and construct the sequence

$$\left\{\left\{x_j^1(1)\otimes\cdots\otimes x_j^m(1)\right\}_{j=1}^{n(1)},\left\{\frac{x_j^1(4)}{4^{1/p}}\otimes x_j^2(4)\otimes\cdots\otimes x_j^m(4)\right\}_{j=1}^{n(4)},...\right\}$$

This sequence is weakly $p-$summable because for all $\phi\in B_{\mathcal{L}^m}$,

$$\sum_{j=1}^{n(1)}\left|\phi\left(x_j^1(1),...,x_j^m(1)\right)\right|^p+\frac{1}{4}\sum_{j=1}^{n(4)}\left|\phi\left(x_j^1(4),...,x_j^m(4)\right)\right|^p+\cdots\leq\sum_{i=1}^{\infty}\frac{1}{i^2}<\infty$$

but

$$\sum_{j=1}^{n(1)}\left\|T\left(x_j^1(1),...,x_j^m(1)\right)\right\|^p+\frac{1}{4}\sum_{j=1}^{n(4)}\left\|T\left(x_j^1(4),...,x_j^m(4)\right)\right\|^p+\cdots\geq 1+1+1+\cdots$$

and that contradicts (iii).■

Observe that the infimum of all K that verify the definition is the same as the infimum of all K that verify (ii) of the last Proposition. That infimum will be called the *strongly $p-$summing norm* of T and it will be denoted by $\|T\|_{ss,p}$. The set of all strongly $p-$summing operators $T\in\mathcal{L}(E_1,...,E_m;F)$ will be denoted by $\mathcal{L}_{ss}^p(E_1,...,E_m;F)$. We have the following straightforward result.

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Proposition 1.3 *(i)* $\left(\mathcal{L}^p_{ss}(E_1,...,E_m;F);\|\cdot\|_{ss,p}\right)$ *is a Banach space.*

(ii) If $T\in\mathcal{L}^1_{ss}(E_1,...,E_m;F)$ *then* $T\in\mathcal{L}^1_{as}(E_1,...,E_m;F)$ *and* $\|T\|_{as,1}\leq\|T\|_{ss,1}$.

We omit the easy proof. We just want to point out that, for (ii), we use the result of [1] that states that $\mathcal{L}^1_{as}(E_1,...,E_m)$ and $\mathcal{L}(E_1,...,E_m)$ are isometrically isomorphic.

The next proposition follows immediately from (ii) of Proposition 1.2, as in the linear case.

Proposition 1.4 *If* $1\leq p<q<\infty$, *then* $\mathcal{L}^p_{ss}(E_1,...,E_m;F)\subset\mathcal{L}^q_{ss}(E_1,...,E_m;F)$ *and* $\|T\|_{ss,q}\leq\|T\|_{ss,p}$.

Observe that the above result is not valid for absolutely summing multilinear mappings. For example, the bilinear mapping $\phi:\ell_2\times\ell_2\to\mathbb{R}$, $\phi(x,y)=\langle x,y\rangle$ is absolutely 1−summing (because every scalar-valued multilinear mapping it is so) but it is not absolutely 2−summing, since the vector unit basis $(e_n)_{n\in\mathbb{N}}$ is weakly 2−summable and $(\phi(e_n,e_n))_{n\in\mathbb{N}}$ is not strongly 2−summable.

The next proposition and its proof is similar to the linear case.

Proposition 1.5 *Let* $T\in\mathcal{L}(E_1,...,E_m;F)$, $R\in\mathcal{L}(F;G)$, $S_i\in\mathcal{L}(X_i;E_i)$, $i=1,...,m$.

(i) If T *is strongly p−summing,* $R\circ T$ *is strongly p−summing and* $\|R\circ T\|_{ss,p}\leq\|R\|\cdot\|T\|_{ss,p}$.

(ii) If T *is strongly p−summing,* $T\circ(S_1,...,S_m)$ *is strongly p−summing and* $\|T\circ(S_1,...,S_m)\|_{ss,p}\leq\|T\|_{ss,p}\cdot\|S_1\|\cdots\|S_m\|$.

(iii) If R *is absolutely p−summing, then* $R\circ T$ *is strongly p−summing and* $\|R\circ T\|_{ss,p}\leq\|R\|_{as,p}\cdot\|T\|$.

(iv) If the m−linear operator $S\in\mathcal{L}(X_1,...,X_m;E_1\widehat{\otimes}\cdots\widehat{\otimes}E_m)$ *defined by* $S\left(x^1,...,x^m\right)=S_1(x^1)\otimes\cdots\otimes S_m(x^m)$ *is strongly p−summing, then* $T\circ(S_1,...,S_m)$ *is strongly p−summing and* $\|T\circ(S_1,...,S_m)\|_{ss,p}\leq\|T\|\cdot\|S\|_{ss,p}$.

Examples 1.6 The proofs of all these examples are similar to the analogous in the linear case.

1) Let K be a compact Hausdorff space, let μ be a positive regular Borel measure on K and let $1\leq p<\infty$. Each $f\in L_p(\mu)$ defines an m−linear multiplication operator $T_f\in\mathcal{L}(^mC(K);L_p(\mu))$, $T_f(g^1,...,g^m)=f\cdot g^1\cdots g^m$. This map is strongly p−summing and $\|T_f\|_{ss,p}=\|f\|_{L_p(\mu)}$.

2) Let $1\leq p<\infty$. Each $\lambda\in\ell_p$ induces an m−linear diagonal operator $D_\lambda\in\mathcal{L}(^m\ell_\infty;\ell_p)$, $D_\lambda\left(x^1,...,x^m\right)=\left(\lambda_n x^1_n\cdots x^m_n\right)_{n\in\mathbb{N}}$ which is strongly p−summing with $\|D_\lambda\|_{ss,p}=\|\lambda\|_{\ell_p}$.

3) Let $1\leq p<\infty$. Any integral m−linear operator $T\in\mathcal{L}_I(E_1,...,E_m;F)$ is strongly p−summing and $\|T\|_{ss,p}\leq\|T\|_I$.

4) Let $1\leq p<\infty$ and (Ω,Σ,μ) be a measure space. For each $f\in L_p(\mu;\mathcal{L}(E_1,...,E_m))$ the m−linear operator $T_f\in\mathcal{L}(E_1,...,E_m;L_p(\mu))$, defined by $T_f\left(x^1,...,x^m\right)=f(\cdot)\left(x^1,...,x^m\right)$, is strongly p−summing and $\|T_f\|_{ss,p}\leq\|f\|_{L_p(\mu;\mathcal{L}(E_1,...,E_m))}$.

5) Any $T\in\mathcal{L}(^m\ell_1;\ell_2)$ is strongly 1−summing (and so strongly p−summing, for every $p\geq1$).

We have some necessary and sufficient conditions of strongly p−summing multilinear operators with domain the space of null sequences c_0.

Proposition 1.7 *Let $T \in \mathcal{L}({}^m c_0; F)$ and let $(e_n)_{n\in\mathbb{N}}$ be the vector unit basis of c_0.*

(i) For any $1 \leq p < \infty$, if T is strongly $p-$summing then $\{\|T(e_n, ..., e_n)\|\}_{n\in\mathbb{N}} \in \ell_p$.

(ii) If $\{\|T(e_{j_1}, ..., e_{j_m})\|\}_{j_1,...,j_m\in\mathbb{N}} \in \ell_1$, then T is strongly $1-$summing.

(iii) If T is 'diagonal' (i.e. $T\left(x^1, ..., x^m\right) = \sum_{j=1}^{\infty} T(e_j, ..., e_j)x_j^1 \cdots x_j^m$) then T is strongly $1-$summing if and only if $\{\|T(e_n, ..., e_n)\|\}_{n\in\mathbb{N}} \in \ell_1$.

PROOF. (i) By [3], $\{e_n \otimes \cdots \otimes e_n\}_{n\in\mathbb{N}}$ is weakly $1-$summable (and so weakly $p-$summable) in $c_0\widehat{\otimes}\cdots\widehat{\otimes}c_0$.

(ii) Let´s denote by $\phi_{j_1,...,j_m}$ the $m-$linear form in c_0 defined by $\phi_{j_1,...,j_m}\left(x^1, ..., x^m\right) = x_{j_1}^1 \cdots x_{j_m}^m$. Then

$$\begin{aligned}
\sum_{l=1}^{n} \left\|T\left(x_l^1, ..., x_l^m\right)\right\| &\leq \sum_{l=1}^{n} \sum_{j_1,...,j_m=1}^{\infty} \|T(e_{j_1}, ..., e_{j_m})\| \left|\phi_{j_1,...,j_m}\left(x_l^1, ..., x_l^m\right)\right| \\
&= \sum_{j_1,...,j_m=1}^{\infty} \|T(e_{j_1}, ..., e_{j_m})\| \sum_{l=1}^{n} \left|\phi_{j_1,...,j_m}\left(x_l^1, ..., x_l^m\right)\right| \\
&\leq \sum_{j_1,...,j_m=1}^{\infty} \|T(e_{j_1}, ..., e_{j_m})\| \sup_{\phi\in B_{\mathcal{L}({}^m c_0)}} \sum_{l=1}^{n} \left|\phi\left(x_l^1, ..., x_l^m\right)\right|
\end{aligned}$$

(iii) It is a consequence of (i) and (ii).■

Remarks 1.8 1) We can generalize Proposition 1.7 (i) to $m-$linear operators $T \in \mathcal{L}({}^m\ell_p; F)$ when $p > m$. Zalduendo [10, Cor. 1] proved that the sequence $\{e_n \otimes \cdots \otimes e_n\}_{n\in\mathbb{N}}$ is weakly $\frac{p}{p-m}-$summable in $\ell_p\widehat{\otimes}\cdots\widehat{\otimes}\ell_p$ (and so it is weakly $q-$summable for every $q \geq \frac{p}{p-m}$). Thus, if $q \geq \frac{p}{p-m}$, T strongly $q-$summing implies $\{\|T(e_n, ..., e_n)\|\}_{n\in\mathbb{N}} \in \ell_q$.

2) Proposition 1.7 (ii) can be stated for any $T \in \mathcal{L}(E_1, ..., E_m; F)$, where $E_1, ..., E_m$ are Banach spaces with Schauder bases.

3) From Proposition 1.7 (i), if $1 \leq p < q < \infty$ and $\lambda \in \ell_q \setminus \ell_p$, the $m-$linear diagonal operator $D_\lambda \in \mathcal{L}({}^m c_0; \ell_q)$, $D_\lambda\left(x^1, ..., x^m\right) = \left(\lambda_n x_n^1 \cdots x_n^m\right)_{n\in\mathbb{N}}$ is strongly $q-$summing but not strongly $p-$summing. This example shows that the inclusions of Proposition 1.4 are strict.

Recall that every $m-$linear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ has an associated linear operator $\widetilde{T} \in \mathcal{L}(E_1\widehat{\otimes}\cdots\widehat{\otimes}E_m; F)$. It is clear that if $\widetilde{T}$ is absolutely $p-$summing, then T is strongly $p-$summing, because the condition for $\widetilde{T}$ to be absolutely $p-$summing is the same inequality as for T to be strongly $p-$summing but for more elements (we have to consider any element of the projective tensor product, not just the elementary tensors).

Open Problem: Does $T \in \mathcal{L}(E_1, ..., E_m; F)$ strongly $p-$summing imply $\widetilde{T} \in \mathcal{L}(E_1\widehat{\otimes}\cdots\widehat{\otimes}E_m; F)$ absolutely $p-$summing?

We want to point out that all the examples of strongly $p-$summing multilinear operators stated above verify that their associated linear operators are absolutely $p-$summing. If the answer of the problem is yes, we can define commutative diagrams and factorizations of strongly $p-$summing $m-$linear operators as in the linear case. We also obtain, in the case of an affirmative answer, that every strongly $p-$summing $m-$linear operator is weakly compact. Thus, in the case of a negative answer we can consider the following related problem.

Related Problem: Does $T \in \mathcal{L}(E_1, ..., E_m; F)$ strongly $p-$summing imply T weakly compact?

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2. The Aron-Berner extension

We need a small review about Aron-Berner extensions of multilinear operators. Let $T \in \mathcal{L}(E_1, ..., E_m; F)$. We will denote by $\overline{T} \in \mathcal{L}(E_1'', ..., E_m''; F'')$ the Aron-Berner extension of T chosen in the following order:

$$\overline{T}\left(z^1, ..., z^m\right) = \overline{z}^1 \circ \cdots \circ \overline{z}^m(T)$$

where

$$\begin{array}{rccc} \overline{z}^j: & \mathcal{L}(E_1, ..., E_j; F'') & \longrightarrow & \mathcal{L}(E_1, ..., E_{j-1}; F'') \\ & \left\langle \overline{z}^j(S)\left(x^1, ..., x^{j-1}\right), f' \right\rangle & = & \left\langle z^j, f' \circ S_{x^1,...,x^{j-1}} \right\rangle \end{array}$$

for every $z^j \in E_j''$, $f' \in F'$ and $S \in \mathcal{L}(E_1, ..., E_j; F'')$ (by $S_{x^1,...,x^{j-1}}$ we mean the linear operator defined by $S_{x^1,...,x^{j-1}}(x) = S\left(x^1, ..., x^{j-1}, x\right)$).

Remarks 2.1 1) We can be more precise for the first step done, that is $\overline{z}^m(T)$. Since $T \in \mathcal{L}(E_1, ..., E_m; F)$ (i.e. its range is F instead of F'') we have the following equality:

$$\left\langle \overline{z}^m(T)\left(x^1, ..., x^{m-1}\right), f' \right\rangle = \left\langle z^m, f' \circ T_{x^1,...,x^{m-1}} \right\rangle = \left\langle \left(T_{x^1,...,x^{m-1}}\right)''(z^m), f' \right\rangle$$

and thus for every $\left(x^1, ..., x^{m-1}\right) \in E_1 \times \cdots \times E_{m-1}$,

$$\overline{z}^m(T)\left(x^1, ..., x^{m-1}\right) = \left(T_{x^1,...,x^{m-1}}\right)''(z^m)$$

2) Suppose that the range of $\overline{z}^m(T)$ is F (so we have $\overline{z}^m(T) \in \mathcal{L}(E_1, ..., E_{m-1}; F)$). In this case, for all $\left(z^1, ..., z^m\right) \in E_1'' \times \cdots \times E_m''$,

$$\overline{T}\left(z^1, ..., z^m\right) = \overline{\overline{z}^m(T)}\left(z^1, ..., z^{m-1}\right)$$

3) For simplicity, we shall consider always the Aron-Berner extension $\overline{T}$ defined above. However, the theorem of this section is also true for Aron-Berner extensions in any order.

A central point in the topic of Aron-Berner extensions for multilinear operators is giving conditions to assure that the extension is actually an extension. If $T \in \mathcal{L}(E_1, ..., E_m; F)$, an extension to the second dual of the domain spaces should be an $m-$linear operator belonging to $\mathcal{L}(E_1'', ..., E_m''; F)$ (and not to $\mathcal{L}(E_1'', ..., E_m''; F'')$). This is what happened with the Aron-Berner extension of strongly $p-$summing multilinear operators. Before stating the result, we want to observe that this property is not true in general for absolutely $p-$summing multilinear operators. Alencar and Matos [1] proved that if $E_1, ..., E_m$ have the Orlicz property, then for every Banach space F, every $T \in \mathcal{L}(E_1, ..., E_m; F)$ is absolutely $1-$summing. On the other hand, if we define the canonical $m-$linear operator $\theta : E_1 \times \cdots \times E_m \longrightarrow E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_m$, $\theta\left(x^1, ..., x^m\right) = x^1 \otimes \cdots \otimes x^m$, it is easy to see that the range of $\overline{\theta}$ is contained in $E_1 \widehat{\otimes} \cdots \widehat{\otimes} E_m$ if and only if all the spaces $E_1, ..., E_m$ are reflexive. Since $L_1[0,1]$ is a non reflexive Banach space with the Orlicz property, it follows that the corresponding mapping $\theta : L_1[0,1] \times \cdots \times L_1[0,1] \longrightarrow L_1[0,1] \widehat{\otimes} \cdots \widehat{\otimes} L_1[0,1]$ is absolutely $1-$summing but its Aron-Berner extension $\overline{\theta}$ is not a proper extension.

Theorem 2.2 *If $T \in \mathcal{L}(E_1, ..., E_m; F)$ is strongly $p-$summing then the Aron-Berner extension of T, $\overline{T}$, belongs to $\mathcal{L}(E_1'', ..., E_m''; F)$.*

PROOF. We will proceed by induction. If $m=1$, T is a linear absolutely $p-$summing operator and since it is weakly compact, $\overline{T}=T''\in\ \mathcal{L}(E'';F)$. Let $m>1$.

First step: We will prove that for all $z^m\in E_m''$, $\overline{z}^m(T)\in\mathcal{L}(E_1,...,E_{m-1};F)$. For that, let´s see that for each $\left(x^1,...,x^{m-1}\right)\in E_1\times\cdots\times E_{m-1}$, $T_{x^1,...,x^{m-1}}$ is an absolutely $p-$summing linear operator. We can consider $\left\|x^i\right\|\leq 1$ for all i. From

$$\begin{aligned}\sum_{j=1}^{n}\left\|T_{x^1,...,x^{m-1}}(x_j^m)\right\|^p &= \sum_{j=1}^{n}\left\|T(x^1,...,x^{m-1},x_j^m)\right\|^p\leq\|T\|_{ss,p}^p\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{n}\left|\phi(x^1,...,x^{m-1},x_j^m)\right|^p\\ &= \|T\|_{ss,p}^p\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{n}\left|\phi_{x^1,...,x^{m-1}}(x_j^m)\right|^p\leq\|T\|_{ss,p}^p\sup_{\gamma_m\in E_m'}\sum_{j=1}^{n}\left|\gamma_m(x_j^m)\right|^p\end{aligned}$$

we obtain that $T_{x^1,...,x^{m-1}}$ is absolutely $p-$summing and that implies that is weakly compact. Then $\left(T_{x^1,...,x^{m-1}}\right)''\in\mathcal{L}(E_m'';F)$ and so, by the first remark above, $\overline{z}^m(T)\in\mathcal{L}(E_1,...,E_{m-1};F)$.

Second step: We will see that $\overline{z}^m(T)$ is strongly $p-$summing, for every $z^m\in E_m''$. Again, it is enough to consider $z^m\in B_{E_m''}$. Given $\left(x_j^1,...,x_j^{m-1}\right)\in E_1\times\cdots\times E_{m-1}$, there exists $f_j'\in B_{F'}$ such that

$$\sum_{j=1}^{n}\left\|\overline{z}^m(T)\left(x_j^1,...,x_j^{m-1}\right)\right\|^p=\sum_{j=1}^{n}\left[f_j'\left(\overline{z}^m(T)\left(x_j^1,...,x_j^{m-1}\right)\right)\right]^p=\sum_{j=1}^{n}\left[f_j'\left(\left(T_{x_j^1,...,x_j^{m-1}}\right)''(z^m)\right)\right]^p$$

Let $\left(x_\alpha^m\right)_\alpha\subset B_{E_m}$such that $x_\alpha^m\xrightarrow{w^*}z^m$. Then

$$f_j'\left(T\left(x_j^1,...,x_j^{m-1},x_\alpha^m\right)\right)\underset{\alpha\to\infty}{\longrightarrow}f_j'\left(\left(T_{x_j^1,...,x_j^{m-1}}\right)''(z^m)\right)$$

and so

$$\sum_{j=1}^{n}\left\|\overline{z}^m(T)\left(x_j^1,...,x_j^{m-1}\right)\right\|^p=\lim_{\alpha\to\infty}\sum_{j=1}^{n}\left[f_j'\left(T\left(x_j^1,...,x_j^{m-1},x_\alpha^m\right)\right)\right]^p.$$

For each α, we obtain

$$\begin{aligned}\sum_{j=1}^{n}\left[f_j'\left(T\left(x_j^1,...,x_j^{m-1},x_\alpha^m\right)\right)\right]^p &\leq \sum_{j=1}^{n}\left\|T\left(x_j^1,...,x_j^{m-1},x_\alpha^m\right)\right\|^p\\ &\leq \|T\|_{ss,p}\sup_{\phi\in B_{\mathcal{L}^m}}\sum_{j=1}^{n}\left|\phi\left(x_j^1,...,x_j^{m-1},x_\alpha^m\right)\right|^p\\ &\leq \|T\|_{ss,p}\sup_{\psi\in B_{\mathcal{L}^{m-1}}}\sum_{j=1}^{n}\left|\psi\left(x_j^1,...,x_j^{m-1}\right)\right|^p\end{aligned}$$

Thus, $\overline{z}^m(T)\in\mathcal{L}_{ss}^p(E_1,...,E_{m-1};F)$. By inductive hypothesis, $\overline{\overline{z}^m(T)}\in\mathcal{L}(E_1'',...,E_{m-1}'';F)$ and by the second remark above, $\overline{T}\left(z^1,...,z^m\right)=\overline{\overline{z}^m(T)}\left(z^1,...,z^{m-1}\right)\in F$. ■

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3. Strongly $p-$summing polynomials

According to the definition of strongly $p-$summing multilinear operators, we can produce a related concept for polynomials.

Definition 3.1 *A polynomial $P \in \mathcal{P}(^m E;F)$ is* **strongly $p-$summing** *if there exists a constant $K \geq 0$ such that for every $x_1, ..., x_n \in E$, we have*

$$\left(\sum_{j=1}^{n} \|P(x_j)\|^p\right)^{\frac{1}{p}} \leq K \sup_{q \in B_{\mathcal{P}(^m E)}} \left(\sum_{j=1}^{n} |q(x_j)|^p\right)^{\frac{1}{p}}$$

As in the multilinear case we have the following equivalences of the definition (we omit the proof because is similar to the previous one):

Proposition 3.2 *Let $P \in \mathcal{P}(^m E;F)$. The following assertions are equivalent:*

(i) P is strongly $p-$summing.

(ii) There exist a regular probability measure μ on $\left(B_{\mathcal{P}(^m E)}, w^*\right)$ and a constant $K \geq 0$ such that for every $x \in E$ we have

$$\|P(x)\| \leq K \left(\int_{B_{\mathcal{P}(^m E)}} |q(x)|^p \, d\mu(q)\right)^{\frac{1}{p}}$$

(iii) If the sequence $\{x_n \otimes \cdots \otimes x_n\}_{n\in\mathbb{N}}$ is weakly $p-$summable in $\widehat{\otimes}_{m,s}E$, then the sequence $\{P(x_n)\}_{n\in\mathbb{N}}$ is strongly $p-$summable in F.

We will denote by $\mathcal{P}^p_{ss}(^m E;F)$ the space of strongly $p-$summing $m-$homogeneous polynomials from E into F. For $P \in \mathcal{P}^p_{ss}(^m E;F)$, we define $\|P\|_{ss,p}$ as the infimum of all constants K that verify the definition (or, equivalently, that verify (ii) of the previous proposition).

Again, we have that $\left(\mathcal{P}^p_{ss}(^m E;F), \|\cdot\|_{ss,p}\right)$ is a Banach space and that every strongly $1-$summing $m-$homogeneous polynomial P is absolutely $1-$summing with $\|P\|_{as,1} \leq \|P\|_{ss,1}$.

A reformulation of Propositions 1.4 and 1.5 for polynomials instead of multilinear operators also holds. And there are similar examples as those in the multilinear case.

Given $P \in \mathcal{P}(^m E;F)$ we denote by $\overset{\vee}{P}$ the symmetric $m-$linear operator associated to P. It is clear that if $\overset{\vee}{P}$ is strongly $p-$summing then P is strongly $p-$summing. Unfortunately, the reverse does not hold as it is shown in the next example.

Before the concrete example and related to this, we state a result about strongly $p-$summing $2N-$homogeneous polynomials in ℓ_{2N}.

Proposition 3.3 *For every Banach space F and every positive integer N, $\mathcal{P}^p_{ss}(^{2N}\ell_{2N};F)$ and $\mathcal{P}(^{2N}\ell_{2N};F)$ are isometrically isomorphic.*

Proof. We can define $q \in \mathcal{P}(^{2N}\ell_{2N})$ as $q(x) = \|x\|^{2N}$ (recall that we consider real Banach spaces) and $\|q\| = 1$. Then, if $P \in \mathcal{P}(^{2N}\ell_{2N};F)$ and $x_1, ..., x_n \in \ell_{2N}$, we have

$$\sum_{j=1}^{n} \|P(x_j)\|^p \leq \|P\|^p \sum_{j=1}^{n} \|x_j\|^{2Np} = \|P\|^p \sum_{j=1}^{n} |q(x_j)|^p$$

and so P is strongly $p-$summing and $\|P\|_{ss,p} \leq \|P\|$. Since the other inequality always holds, we have an isometry. ■

Example 3.4 *A* $2-$ *homogeneous polynomial* P *which is strongly* $1-$*summing but its associated symmetric bilinear operator* $\overset{\vee}{P}$ *is not strongly* $1-$*summing.*

Let $P : \ell_2 \longrightarrow \ell_2\widehat{\otimes}\ell_2$, $P(x) = x \otimes x$. By the previous proposition, P is strongly $1-$summing. However, the symmetric bilinear operator $\overset{\vee}{P}$ is not strongly $1-$summing. Indeed, observe that $\overset{\vee}{P}$: $\ell_2 \times \ell_2 \longrightarrow \ell_2\widehat{\otimes}\ell_2$ is defined by

$$\overset{\vee}{P}(x,y) = \frac{x \otimes y + y \otimes x}{2}$$

If $\overset{\vee}{P}$ were strongly $1-$summing, by the proof of Theorem 2.2, for each $x \in \ell_2$, the linear operator $\left(\overset{\vee}{P}\right)_x$ should be absolutely $1-$summing and so completely continuous. But $e_n \xrightarrow{w} 0$ and $\left\|\overset{\vee}{P}(x,e_n)\right\| \geq \frac{\|x\|}{4}$ for every $n \geq n_0$.

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