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Equilibrium Portfolios in the Neoclassical Growth Model*

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Abstract

This paper studies equilibrium portfolios in the standard neoclassical growth model under uncertainty with heterogeneous agents and dynamically complete markets. Preferences are purposely restricted to be quasi-homothetic. The main source of heterogeneity across agents is due to different endowments of shares of the representative firm at date 0. Fixing portfolios is the optimal strategy in stationary endowment economies with dynamically complete markets. Whenever an environment displays changing degrees of heterogeneity across agents, the trading strategy of fixed portfolios cannot be optimal in equilibrium. Very importantly, our framework can generate changing heterogeneity if and only if either minimum consumption requirements are not zero or labor income is not zero and the value of human and non-human wealth are linearly independent.

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1 Introduction

This paper studies equilibrium portfolios in the standard one-sector neoclassical growth model under uncertainty with heterogeneous agents. There is an aggregate technology to produce the unique consumption good which is either consumed or invested. This technology displays constant returns to scale, is subject to productivity shocks and is operated by a representative firm. Preferences are purposely restricted such that momentary utility functions are quasi-homothetic. This implies that Engel curves are affine linear in lifetime human and non-human wealth. Heterogeneity across agents can arise due to differences in initial wealth and preferences in the following way. First, agents may have different shares of the representative firm at date 0 but they have the same labor income profile. That is, they can differ in their non-human wealth but share the same human wealth. Secondly, agents may also have different preferences within the class mentioned before due to different minimum consumption requirements. We abstract from any kind of friction and, in particular, we assume that markets are dynamically complete. The environment is consequently an extension of Brock and Mirman's [1972] seminal contribution in the spirit of Chatterjee's [1994] pioneering work. The main differences with Chatterjee [1994] are that we consider a stochastic environment due to aggregate technology shocks, agents have labor income and minimum consumption requirements can differ among agents.

Preferences were purposely restricted to the class mentioned to minimize both the incentives and the possibilities for risk sharing. Exploiting some features of this preference representation, the analysis shows how to explicitly determine equilibrium portfolios as a function of preference parameters, initial endowments and different sources of wealth.

Recently, the ability of the celebrated Lucas [1978] tree model to generate non-trivial asset trading has been under scrutiny. The original version is silent about the evolution of equilibrium portfolios since they are kept fixed in the representative agent framework. However, Judd, Kubler and Schmedders [2003] (JKS from now on) study the stationary Lucas tree model with complete markets under general assumptions regarding heterogeneity across agents. Their surprising finding is that, after some initial rebalancing in short and long-lived assets, agents choose a fixed equilibrium portfolio which is independent of the aggregate state of economy. Unarguably, the Lucas tree model is one of the most popular asset pricing models in modern finance theory and this equilibrium property questions its full further applicability.[1] JKS conclude that some friction (informational, financial, etc.) must play a significant role in generating nontrivial asset trading in that framework.

Espino and Hintermaier [2005] cast some doubts on the necessity of those frictions by studying the production economy proposed by Brock [1982] with heterogeneous agents under dynamically complete markets. Production of the consumption good is carried out by several neoclassical firms subject to idiosyncratic productivity shocks. They show that their environment can generate a nontrivial amount of equilibrium asset trading resulting from portfolios that in general depend upon the aggregate state of the economy. In other words, the trading strategy of fixed portfolios is not optimal in equilibrium. In their economy, the time dependence of agents' heterogeneity is determined by the evolution of the distribution of capital across firms. This is mainly due to the fact that agents have different labor productivities across these different production units.

A recent independent paper by Bossaerts and Zame [2005] has also discussed JKS's interpretation of their result. They assume that individual endowments are not stationary even though this last property holds at the aggregate level. In that case, they construct examples where equilibrium portfolios are not kept fixed. But after all, assuming away stationarity of individual endowments implies that the dis-

[1] Mehra and Prescott [1985] point out another unsatisfactory feature of the model in relation to its asset pricing implications.

tribution of the aggregate endowment is nonstationary and consequently the degree of heterogeneity across agents is naturally changing.

The crucial aspect pointed out by Espino and Hintermaier [2005] and, implicitly, by Bossaerts and Zame [2005] is the following. Whenever the environment under study generates changing degrees of heterogeneity across agents, the trading strategy of fixed portfolios will not in general be optimal in equilibrium. The second of these papers simply assumes that a crucial dimension of heterogeneity changes through time. On the other hand, one might suspect that the results in the first of these papers rely on the particular assumptions regarding different labor productivities across firms (what they call *limited labor substitutability)*.

This paper endogenously generates changing degrees of heterogeneity abstracting from all those "frictions". We study the simplest extension of the Lucas tree model to a production economy where we are able to analyze asset trading isolating the impact of capital accumulation on the evolution of equilibrium portfolios. This setting is the stochastic one-sector optimal growth model with heterogeneous agents. Very importantly, we show that this framework can generate changing heterogeneity if and only if either minimum consumption requirements are not zero or labor income is not zero and the value of human and non-human wealth are linearly independent.

At date 0 agents can be different due to two reasons in our setting. First, they might have different minimum consumption requirements. This kind of heterogeneity is kept fixed as time and uncertainty unfold. Secondly, agents can own different shares of aggregate non-human wealth at date 0, which is given by the initial value of the firm. If equilibrium evolves such that these shares are kept fixed at the initial level, this crucial second dimension of heterogeneity across agents also remains unchanged. Consequently when both channels of heterogeneity remain unchanged, agents can trade away individual risk through fixed portfolios.

In our framework, each agent's participation in non-human wealth is not constant in equilibrium either if minimum consumption requirements are not zero or if labor income is not zero and human and non-human wealth are linearly independent. Consider the case where there are no minimum consumption requirements. Under these preference representations, complete markets will imply that each agent owns a fixed share of aggregate total wealth, which includes human and non-human wealth. Both levels of wealth depend on the evolution of the stock of capital and thus equilibrium portfolios will adjust accordingly to keep these shares fixed whenever human and non-human wealth are not perfectly linearly correlated.

When there is no labor income but minimum consumption requirements are not zero, the intuition is similar. In this case, agents will keep a fixed fraction of aggregate non-human wealth net of the value of the aggregate minimum consumption requirement. This will imply that equilibrium portfolios determining individual wealth cannot be kept fixed and independent of the state of the economy.

In recent independent work, Obiols-Homs and Urrutia [2005] study the evolution of asset holdings in a deterministic version of our framework with log preferences. They find sufficient conditions such that the coefficient of variation in assets across agents decreases monotonically along the transition to the steady state from below. In contrast, Chatterjee [1994] studies the evolution of wealth in an economy without labor income. He shows that if the economy is growing to the steady state, wealth inequality changes monotonically along the transition whenever minimum consumption requirements are not zero.

This paper then extends and complements the existing related literature mentioned above in nontrivial dimensions. It is the natural framework to directly compare our results with the literature studying asset trading volume and thus to investigate the implications of capital accumulation on the evolution of equilibrium portfolios. It shows that changing distributions of aggregate wealth studied by Bossaerts and Zame [2005] can be the natural consequence of capital accumulation, even in simple environments like ours. Moreover, we provide a relatively simple method to solve for equilibrium portfolios in the standard neoclassical growth model under alternative complete market structures. Since this framework was the natural benchmark to study quantitatively the behavior of different asset prices, it can also be used as a benchmark to evaluate quantitatively its predictions about the evolution of asset holdings, stock trading volume, etc.

The paper is organized as follows. In Section 2, we describe the economy and characterize the set of Pareto optimal allocations. In Section 3, we study equilibrium portfolios in the corresponding economy with sequential trading and dynamically complete markets. Section 4 concludes. All proofs are in the Appendix.

2 The Economy

We consider an economy populated by I (types of) infinitely-lived agents where $i \in \{1, ..., I\}$. Time is discrete and denoted by $t = 0, 1, 2, ...$ Each agent is endowed with one unit of time every period but for simplicity we assume that leisure is not valued. There is only one consumption good which can be either consumed or invested. Goods invested transform one-to-one in new capital next period. There is a constant returns to scale aggregate technology to produce the consumption good operated by a representative firm. This technology is subject to productivity shocks and s_t represents the realization of this shock at date t. We assume that $\{s_t\}$ is a finite state first-order stationary Markov process. Transition probabilities are denoted by $\pi(s, s') > 0$ where $s_t, s, s' \in S = \{s_1, ..., s_N\}$. Let $s^t = (s_0, ..., s_t) \in S^{t+1}$ represent the partial history of aggregate shocks up to date t. We write s^{t+1}/s^t to denote that s^{t+1} is an immediate successor of s^t. These histories are observed by all the agents. The probability of s^t is constructed from π in the standard way and $x(s^t)$ denotes the value of x at the node s^t.

A consumption bundle for agent i is a sequence of functions $\{c_t\}_{t=0}^{\infty}$ such that $c_t : S^{t+1} \to [\gamma_i, +\infty)$ for all t and $\sup_{s^t} c(s^t) < \infty$. $\mathbf{C}_i$ is agent i's consumption set with all these sequences as elements. The parameter $\gamma_i \geq 0$ is interpreted as the minimum consumption level required by agent i. Agent i's preferences on $\mathbf{C}_i$ are represented by expected, time-separable, discounted utility. That is , if $c_i \in \mathbf{C}_i$ then:

$$U(c_i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) u_i(c_i(s^t)),$$

where $\beta \in (0,1)$ and the momentary utility function u_i belongs to the following class:

$$u_i(c) = \begin{cases} \frac{(c-\gamma_i)^{1-\sigma}}{1-\sigma} & \text{if } \sigma \neq 1 \text{ and } \sigma > 0 \\ \ln(c-\gamma_i) & \text{if } \sigma = 1 \end{cases},$$

where $(c - \gamma_i) \geq 0$ (with strict inequality if $\sigma \geq 1$).

Let $K(s^t) \geq 0$ denote the stock of capital chosen by the representative firm at the node s^t and available in period $t+1$ to produce with the aggregate technology. The depreciation rate is given by $\delta \in (0,1)$. Let $sF(K,L)$ represent this technology, where K is the stock of capital available, L is the level of labor and s is the productivity shock. $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is homogeneous of degree 1, strictly concave, strictly increasing in K, weakly increasing in L and continuously differentiable.[2] We also assume that for all $L > 0$: (a) $\partial F(0,L)/\partial K = \infty$ and (b) $\lim_{K\to\infty} \partial F(K,L)/\partial K = 0$. Condition (a) rules out corner solutions for capital accumulation. More importantly, condition (b) guaranties that there exists some $(\underline{K}, \overline{K})$ such that $0 \leq \underline{K} \leq K(s^t) \leq \overline{K}$ for all s^t since consumption must be bounded from below.[3] Without loss of generality, we can restrict ourselves to $K(s^t) \in X \equiv [\underline{K}, \overline{K}]$ for all s^t. We denote $K = (K(s^t)_{s^t})$ as a capital accumulation path, where $K_0 \in X$ is the initial stock of capital which is assumed to be greater than $\underline{K}$. Note that $L(s^t) = I$ for all s^t and denote $f(K(s^{t-1})) = F(K(s^{t-1}), I)$. Below we will refer to the special case of *unproductive labor* when $F(K,L) = F(K)$ for all L and consequently $F_L(K,L) = 0$ for all (K,L). In this case, we still assume that F is strictly concave.

To determine equilibrium portfolios we proceed as follows. We first characterize the set of Pareto optimal allocations. Under our assumptions about preferences,

[2] Under constant returns to scale, any price taking industry behaves as a single representative price taking firm.

[3] More specifically, $\underline{K}$ solves $\min_{s\in S}(s)F(\underline{K}, I) - \delta\underline{K} = \sum_{i=1}^{I} \gamma_i$ and $\overline{K}$ is the standard upper bound in the neoclassical growth model.

the problem reduces to solve for aggregate variables given the existence of a fictitious *aggregate representative consumer* (ARC). Then, in the next section we proceed studying a competitive market arragenment with sequential trading to analyze the evolution of equilibrium portfolios.

Planner's Problem

Under our concavity assumptions on both utility and the production functions, the set of Pareto optimal allocations can be parametrized by welfare weights.[4] Suppose that α_i is the welfare weight assigned by the planner to agent i. Given a vector of welfare weights $\alpha = (\alpha_i)_{i=1}^I$, the planner's problem is given by:

$$V(s_0, K_0; \alpha) = \max_{(c,K)} \sum_{i=1}^{I} \alpha_i \left\{ \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \frac{(c_i(s^t) - \gamma_i)^{1-\sigma}}{1-\sigma} \right\}, \tag{PP}$$

subject to

$$\begin{aligned} \sum_{i=1}^{I} c_i(s^t) + K(s^t) - (1-\delta)K(s^{t-1}) &= s_t f(K(s^{t-1})) \qquad \text{for all } s^t, \\ c_i &\in \mathbf{C}_i,\ K(s^t) \in X \qquad \text{for all } s^t \text{ and all } i, \\ K^0 &\in X \text{ given.} \end{aligned}$$

Under these preference representations, it is well-known that the solution to (PP) is equivalent to solve:

$$V(s_0, K_0) = \max_{(c,K)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \frac{(C(s^t) - \gamma)^{1-\sigma}}{1-\sigma}, \tag{APP}$$

subject to

$$\begin{aligned} C(s^t) + K(s^t) - (1-\delta)K(s^{t-1}) &= s_t f(K(s^{t-1})) \qquad \text{for all } s^t, \\ C(s^t) - \gamma &\geq 0,\ K(s^t) \in X \qquad \text{for all } s^t, \\ K^0 &\in X \text{ given,} \end{aligned}$$

[4] That is, the utility possibility set is convex and closed and thus we can immediately apply the supporting hyperplane theorem.

where $\gamma = \sum_{i=1}^{I} \gamma_i$ and $C(s^t)$ is aggregate consumption. Given a vector α, optimal individual consumption for each i is determined by:

$$[c_i(s^t;\alpha) - \gamma_i] = \frac{(\alpha_i)^{1/\sigma}}{\sum_{h=1}^{I} (\alpha_h)^{1/\sigma}} [C(s^t) - \gamma]. \tag{1}$$

The equivalence can be easily determined because (1) satisfies the necessary and sufficient first order conditions for the problem (PP) and the solution to (APP) is immune to affine linear transformations of preferences.[5] Very importantly, this implies that the evolution of the stock of capital is independent of the initial distribution of wealth. Note also that (1) implies that the marginal rate of substitution of any agent equals the ARC's marginal rate of substitution.

The former discussion reduces the problem to the traditional one-sector neoclassical growth model under uncertainty. It is a standard exercise to establish that this problem has a recursive formulation.[6] The value function $V : \mathbf{S} \times X \to \mathbb{R}$ solves the following functional equation:

$$V(s,K) = \max_{(C,K')} \left\{ \frac{(C-\gamma)^{1-\sigma}}{1-\sigma} + \beta \sum_{s'} \pi(s,s') V(s',K') \right\}, \tag{RAPP}$$

subject to

$$\begin{aligned} C + K' - (1-\delta)K &= sf(K), \\ C - \gamma &\geq 0,\ K' \in X. \end{aligned}$$

Moreover, it can also be shown that V is strictly increasing, strictly concave in K and continuously differentiable in the interior of X. The solution to the problem (RAPP) is a set of continuous policy functions $(C(s,K), K'(s,K))$. To fully characterize the set of Pareto optimal allocations, we construct the recursive version of

[5] In this case, $\sum_{i=1}^{I} \alpha_i \frac{(c_i(s^t)-\gamma_i)^{1-\sigma}}{1-\sigma}$ reduces to $\frac{(C(s^t)-\gamma)^{1-\sigma}}{1-\sigma} \left(\sum_{i=1}^{I} (\alpha_i)^{1/\sigma} \right)^{\sigma}$ for each s^t.

[6] See Stokey, Lucas and Prescott [1989, Section 10.1]). If $\sigma \geq 1$, utility functions are unbounded from below. To deal with this case, the techniques explained in Alvarez and Stokey [1998] can be adapted. For that, notice that momentary utility functions are homogeneous in $c - \gamma_i$ and the problem can be transformed to the optimal growth model with a production function $sf(k) - \sum \gamma_i$. See in particular Le Van and Morhaim [2002].

individual consumption (1) by:

$$[c_i(s,K,\alpha)-\gamma_i]=\frac{(\alpha_i)^{1/\sigma}}{\sum_{j=1}^{I}(\alpha_j)^{1/\sigma}}[C(s,K)-\gamma], \tag{2}$$

for all (s,K), given a vector of welfare weights α.

Next we will consider competitive decentralization with sequential trading and complete markets to study asset trading and its corresponding equilibrium portfolios.

3 Sequential Trading and Recursive Competitive Equilibrium

We assume the following trading opportunities. Every period t and after having observed s^t, agents meet in spot markets to trade the consumption good and different assets. They can trade firm's shares, where $\theta_i(s^t)$ is the number of shares chosen by agent i at s^t. There is one outstanding share and then $\sum_i \theta_i(s^t)=1$ for all s^t.[7] We do not impose short-selling constraints. We assume that agent i is endowed with $\theta_i(s_{-1})=\theta_i^0>0$ shares of the firm at date 0. It is assumed that for each agent i this share is large enough such that the optimal consumption path is in the interior of the consumption set (see the Appendix for details). Let $p(s^t)$ be the ex-dividend price of one share at s^t. Let $w(s^t)$ be the wage paid by the firm per unit of labor at s^t.

Agents can also trade a complete set of fully enforceable Arrow securities with zero net supply. The Arrow security s' traded at s^t pays one unit of consumption next period if $s^{t+1}=(s^t,s')$ and 0 otherwise. Let $q(s^t)(s')$ be the price of this security at s^t. Denote $a_i(s^t)$ as agent $i's$ holdings of this security at s^t where $a_i(s_0)=0$ for all i. To rule out Ponzi schemes, we restrict agents to bounded trading strategies. These (implicit) bounds are assumed to be sufficiently large such that they do not bind in equilibrium. All these prices are in units of the s^t-consumption good.

The firm chooses labor and accumulates capital to maximize its value. In this framework with sequential trading, if the firm reaches s^t with a stock of capital

[7] Firm's financial policies do not affect equilibrium allocations and prices since the Modigliani-Miller theorem holds in this framework.

$K(s^{t-1})$ then its problem can be expressed:

$$V_F(s^t, K(s^{t-1})) = \max_{K(s^t), L(s^t)} \left\{ d(s^t, K(s^{t-1})) + \sum_{s'} q(s^t)(s') V_F(s^t, s', K(s^t)) \right\}, \quad \text{(FP)}$$

subject to

$$d(s^t, K(s^{t-1})) = s_t F(K(s^{t-1}), L(s^t)) + (1-\delta)K(s^{t-1}) - K(s^t) - w(s^t)L(s^t), \quad (3)$$

for all s^t. Here, $V_F(s^t, K(s^{t-1}))$ is the value of the firm with stock of capital $K(s^{t-1})$ at s^t.

From the consumer side, given a price system $(\mathbf{p}, \mathbf{q}, \mathbf{w})$ agent i's problem is given by:

$$\max_{(c_i, a_i, \theta_i)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t) \frac{(c_i(s^t) - \gamma_i)^{1-\sigma}}{1-\sigma}, \quad \text{(AP)}$$

subject to

$$c_i(s^t) + p(s^t)\theta_i(s^t) + \sum_{s'} q(s^t)(s') a_i(s^t, s') = \phi_i(s^t) + w(s^t), \quad (4)$$

$$\phi_i(s^t, s') = [p(s^t, s') + d(s^t, s')]\theta_i(s^t) + a_i(s^t, s'), \quad (5)$$

where $c_i \in C_i$ and (a_i, θ_i) are bounded. Here, $\phi_i(s^t)$ denotes agent i's non-human wealth at s^t. A competitive equilibrium in this framework is defined in the standard way.

Definition 1 *A **Competitive Equilibrium with Sequential Trading (STCE)** is a price system* $(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}, \widehat{\mathbf{w}})$*, an allocation* $\left\{(\widehat{c}_i)_i, \widehat{K}, \widehat{L}\right\}$ *and equilibrium portfolios* $(\widehat{a}_i, \widehat{\theta}_i)_i$ *such that:*

(STCE 1) *Given* $(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}, \widehat{\mathbf{w}})$, $(\widehat{c}_i, \widehat{\theta}_i, \widehat{a}_i)$ *solves agent i's problem (AP) for each i.*

(STCE 2) *Given* $(\widehat{\mathbf{p}}, \widehat{\mathbf{q}}, \widehat{\mathbf{w}})$, $(\widehat{K}, \widehat{L})$ *solves the firm's problem (FP) where* $\widehat{L}(s^t) = I$ *for all* s^t.

(STCE 3) *All markets clear. For all* s^t :

$$\begin{aligned} \sum_{i=1}^{I} c_i(s^t) + [K(s^t) - (1-\delta)K(s^{t-1})] &= s_t f(K(s^{t-1})), \\ \sum_{i=1}^{I} \widehat{a}_i(s^t, s') &= 0 \text{ for all } s', \\ \sum_{i=1}^{I} \widehat{\theta}_i(s^t) &= 1. \end{aligned}$$

We can establish the affine linearity of individual consumption with respect to total wealth. This is a direct consequence of the particular preference representation that we have purposely assumed to minimize asset trading incentives and it will be an important aspect to simplify the analysis of our main result.

We can summarize the previous discussion as follows. Let $V_w(s^t)$ be the value of human wealth at s^t (e.g. the present discounted value of labor income).[8]

Lemma 2 *There exist* $b(s^t)$ *and* $z(s^t)$ *such that agent i's consumption can be expressed by:*

$$c_i(s^t) = \gamma_i \left(1 - b(s^t)\right) + z(s^t) \left(\phi_i(s^t) + V_w(s^t)\right), \tag{6}$$

for all s^t *and all* i.

The Markovian structure of this economy assures that there exists an equivalent *Recursive Competitive Equilibrium* (RCE) for each STCE.[9] Consider the set of state variables. At the consumer level, it is described by individual wealth, ϕ_i. At the firm level, k describes the firm's stock of capital. Let Φ and K describe the distribution of wealth and the aggregate stock of capital, respectively. Therefore, the set of aggregate state variables is fully described by (s, Φ, K). The price system is given by $\mathbf{p}, \mathbf{Q}, \mathbf{w} : \mathbf{S} \times \mathbb{R}_+^I \times \mathbb{R}_+ \to \mathbb{R}_{++}$, representing prices for shares, Arrow securities and wages, respectively. To simplify notation, we directly impose the labor market equilibrium condition such that $L(s, \Phi, K) = I$ for all (s, Φ, K).

[8] Expression (6) below is a generalization of Chatterjee [1994, equation (4)] for a setting that includes aggregate uncertainty and labor income.

[9] See Ljungqvist and Sargent [2004, Chapter 8 and 12] for a discussion.

Definition 3 *A RCE is a set of value functions for the individuals* (V_i)*, a value function for the firm* (V_F)*, a set of policy functions for the individuals* (c_i, a_i', θ_i')*, a policy function for the firm* (k')*, a set of prices* $(\mathbf{p}, \mathbf{Q}, \mathbf{w})$ *and laws of motion for the aggregate state variables* $\Phi' = G(s, \Phi, K)$ *and* $K' = H(s, \Phi, K)$ *such that:*

(RCE 1) *Given* $(\mathbf{p}, \mathbf{Q}, \mathbf{w})$*, for each agent* i (c_i, a_i', θ_i') *are the corresponding policy functions and* $(V_i, c_i, a_i', \theta_i')$ *solves:*

$$V_i(\phi_i, s, \Phi, K) = \sup_{(c_i, a_i', \theta_i')} \{u(c_i) + \beta E\left[V_i(\phi_i', s', \Phi', K') \parallel s\right]\},$$

subject to

$$c_i + p(s, \Phi, K)\theta_i' + \sum_{s'} Q(s, \Phi, K)(s')a_i'(s') = \phi_i + w(s, \Phi, K),$$

$$\phi_i' = \left[p(s', \Phi', K') + d(s', \Phi', K')\right]\theta_i' + a_i'(s'),$$

where $\Phi' = G(s, \Phi, K)$ *and* $K' = H(s, \Phi, K)$*.*

(RCE 2) *Given* $(\mathbf{p}, \mathbf{Q}, \mathbf{w})$*,* V_F *is the recursive version solving (FP) and* (k') *solves the firm's problem, where dividends are given by:*

$$d(s, k, \Phi, K) = sf(k) + (1 - \delta)k - k'(k, s, \Phi, K) - w(s, \Phi, K)I.$$

(RCE 3) *All markets clear. For all* (ϕ_i, k, s, Φ, K) *:*

$$\sum_i c_i(\phi_i, s, \Phi, K) + (k'(k, s, \Phi, K) - (1 - \delta)k) = sf(k),$$

$$\sum_i a_i'(\phi_i, s, \Phi, K)(s') = 0 \qquad \text{for all } s',$$

$$\sum_i \theta_i'(\phi_i, s, \Phi, K) = 1.$$

(RCE 4) *Consistency. For all* (s, Φ, K) *and each* i *:*

$$\begin{aligned} K' &= H(s, \Phi, K) = k'(s, K, \Phi, K), \\ \Phi_i' &= G_i(s, \Phi, K) = [p(G(s, \Phi, K), H(s, \Phi, K)) \\ &\quad + d(G(s, \Phi, K), H(s, \Phi, K))]\theta_i'(\Phi_i, s, \Phi, K) + a_i'(\Phi_i, s, \Phi, K). \end{aligned}$$

We are interested in decentralizing a particular Pareto optimal allocation as a RCE defined above. With that purpose, we proceed constructively. Consider the policy functions (C, K') for the problem (RPP). Define the stochastic discount factor as follows:

$$Q(s,K)(s') = \beta\pi(s,s')\frac{(C(s',K'(s,K))-\gamma)^{-\sigma}}{(C(s,K)-\gamma)^{-\sigma}}, \tag{7}$$

for all (s, K, s'). Individual consumption is constructed using (2). It will be useful to have a recursive version of (6). The value of human wealth, which is independent of welfare weights α, can be expressed as follows:

$$V_w(s,K) = w(s,K) + \sum_{s'} Q(s,K)(s')V_w(s',K'(s,K)). \tag{8}$$

Also, define:

$$V_P(s,K) = 1 + \sum_{s'} Q(s,K)(s')V_P(s',K'(s,K)), \tag{9}$$

$$V_X(s,K,s') = \frac{C(s',K'(s,K))-\gamma}{C(s,K)-\gamma} + \sum_{z} Q(s',K'(s,K))(z)V_X(s',K'(s,K),z), \tag{10}$$

where $V_X(s,K) = \sum_{s'} Q(s,K)(s')V_X(s,K,s')$. Here, V_P is interpreted as the value of a perpetual bond that pays 1 unit of the consumption good in each period and in each state. V_X is interpreted as the presented discounted value of aggregate consumption growth, net of minimum requirements. It can be shown that each of the previous functional equations have a unique continuous solution V_w, V_P and V_X. See Lemma 10 in the Appendix for technical details.

Individual consumption must thus satisfy the recursive version of (6):

$$c_i(s,K,\alpha) = \gamma_i(1-b(s,K)) + z(s,K)\left(\phi_i(s,K,\alpha) + V_w(s,K)\right), \tag{11}$$

where $b(s,K) = [1+V_X(s,K)]^{-1}V_P(s,K)$ and $z(s,K) = [1+V_X(s,K)]^{-1}$.

Note that (2) and (11) imply that the ARC's consumption is given by:

$$C(s,K) = \gamma\left(1-b(s,K)\right) + z(s,K)\left(V_F(s,K,\alpha) + V_w(s,K)I\right),$$

which is precisely the expression expected to be obtained under these preferences representation for the ARC.

We are ready now to decentralize a Pareto optimal allocation as a RCE with zero initial transfers, conditional upon our assumptions about the initial distribution of shares $\theta^0 = (\theta_i^0)_{i=1}^I$. Let (s_0, K_0) be the date 0 state of the economy. We will follow Negishi's [1960] approach and show that given (s_0, K_0, θ^0) there exists a vector $\alpha^0 = \alpha(s_0, K_0, \theta^0)$ such that the corresponding Pareto optimal allocation can be decentralized as a RCE with zero initial transfers for each agent. From now on, we impose the consistency conditions (RCE 4) and thus we avoid writing policy functions depending on individual state variables.

Note first that we allow for trading in both Arrow securities and shares. This means that there are redundant assets since we have $S+1$ assets with linearly independent payoffs while there are S states of nature. To illustrate our results, we shut down stock trading where we impose $\theta_i'(s, \Phi, K) = \theta_i^0$ for all i and all (s, Φ, K). As discussed below, it is a relatively simple exercise to apply the analytical tools developed here to study and explicitly determine asset trading and equilibrium portfolios in any framework with arbitraries complete market structures.

Define wages and stock prices as follows:

$$w(s, K) = sF_2(K, I), \tag{12}$$

$$p(s, K) = \sum_{s'} Q(s, K)(s')[p(s', K'(s, K)) + d(s', K'(s, K))]. \tag{13}$$

To determine individual wealth, consider the following functional equation constructed from the agent i's budget constraint (22):

$$\phi_i(s, K, \alpha) = c_i(s, K, \alpha) - w(s, K) + \sum_{s'} Q(s, K)(s')\phi_i(s', K'(s, K), \alpha). \tag{14}$$

As mentioned before, both the operator determining the stock price (13) and individual wealth (14) will have unique continuous functions as solutions. We need to show that there exists a unique $\alpha^0 = \alpha(s_0, K_0, \theta^0)$ such that:

$$\phi_i(s_0, K_0, \alpha^0) = \theta_i^0 \left[p(s_0, K_0) + d(s_0, K_0)\right] = \theta_i^0 V_F(s_0, K_0).$$

Consequently, the Pareto optimal allocation corresponding to α^0 can be decentralized as a RCE with zero initial transfers.

We fix α^0 to obtain equilibrium Arrow security holdings as follows. First, define:

$$A_i(s, K, \alpha^0) = \phi_i(s, K, \alpha^0) - \theta_i^0 \left[p(s, K) + d(s, K)\right].$$

Therefore, equilibrium portfolios in this RCE will be given by:

$$\begin{aligned} \theta_i'(s, \Phi, K) &= \theta_i^0, \\ a_i'(s, \Phi, K)(s') &= A_i(s', K'(s, K), \alpha^0), \end{aligned} \tag{15}$$

where, for each i, $\Phi_i = \phi_i(s, K, \alpha^0)$ is uniquely determined by (s, K) given α^0. Note that from (15) we can observe that equilibrium portfolios depend upon (s, K) because they determine next period's stock of capital and thus the evolution of relative individual non-human wealth.

We can now summarize this discussion with the following result where we also show how to explicitly express α^0 and A_i's as functions of V_F, V_w, V_P, θ^0 and γ_i's.

Proposition 4 *There exists a unique welfare weight $\alpha^0 = \alpha(s_0, K_0, \theta^0)$ given by:*

$$\left(\alpha_i(s_0, K_0, \theta^0)\right)^{1/\sigma} = \frac{\theta_i^0 V_F(s_0, K_0) + V_w(s_0, K_0) - V_P(s_0, K_0)\gamma_i}{V_F(s_0, K_0) + V_w(s_0, K_0) I - V_P(s_0, K_0)\gamma}, \tag{16}$$

such that the corresponding Pareto optimal allocation can be decentralized as a RCE. The price system is given by (7), (13) and (12), equilibrium individual consumption by (2) and equilibrium portfolios by (15) where:

$$\begin{aligned} A_i(s, K, \alpha^0) &= [\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0] V_F(s, K) + [\gamma_i - \left(\alpha_i^0\right)^{1/\sigma} \gamma] V_P(s, K) \\ &\quad + [I \left(\alpha_i^0\right)^{1/\sigma} - 1] V_w(s, K). \end{aligned} \tag{17}$$

We say that *the fixed equilibrium portfolio property* is satisfied if for each s', for all (s, Φ, K) and for all i, it follows that $a_i'(s, \Phi, K)(s') = a_i'(s')$. That is, individual portfolios are kept fixed, independently of the aggregate state of the economy. The next result shows that this property holds if we reduce this framework to an endowment economy. Whenever capital is unproductive and $\delta = 0$, we get a version of the stationary Lucas tree model with heterogeneous agents which is a particular version of JKS [2003].

Proposition 5 ***(JKS [2003])*** *In the endowment version of this economy, equilibrium portfolios are fixed such that:*

$$a_i'(s, \Phi)(s') = a_i'(s') \qquad \text{for all } (s, \Phi) \text{ and for all } s',$$

where $\Phi_i = \phi_i(s, \alpha^0)$ *for each* i.

It is important to notice that the demand for Arrow securities is independent of the unique aggregate state variable, s. That is, independently of the state s today (and thus independent of the uniquely determined wealth distribution, $\Phi(s)$), agents choose to construct fixed equilibrium portfolios of Arrow securities. JKS [2003] show how to extend this result in several dimensions for pure exchange economies.

It turns out that if minimum consumption requirements and labor income are *both* assumed away, this environment is still unable to generate changing equilibrium portfolios and thus the fixed equilibrium portfolio property is preserved.

Proposition 6 *If* $\gamma_i = 0$ *for all* i *and* $w(s, K) = 0$ *for all* (s, K)*, then the equilibrium portfolio is kept fixed and given by:*

$$\begin{aligned} \theta_i'(s, \Phi, K) &= \theta_i^0, \qquad (18) \\ a_i'(s, \Phi, K)(s') &= 0, \ \text{for all } s' \text{ and all } (s, \Phi, K). \end{aligned}$$

Now we show that this result might not be robust in the more general setting introduced above. First, if minimum consumption requirements differ from 0 for at least one agent, then the trading strategy of fixed portfolios cannot be optimal in equilibrium even if there is no labor income.

Proposition 7 *Suppose that labor is unproductive with $w(s, K) = 0$ for all (s, K). If $\gamma_i > 0$ for some agent i, then fixed portfolios cannot be optimal in equilibrium.*

Secondly, we show that under certain conditions the introduction of labor income (*productive labor*) is sufficient to generate changing equilibrium portfolios.

Proposition 8 *Assume away minimum consumption requirements where $\gamma_i = 0$ for all i. If $w(s, K) > 0$ for all (s, K), then the fixed equilibrium portfolio property holds only if human wealth and non-human wealth are perfectly linearly correlated.*

It is important to notice that under the assumptions of Proposition 7, the fixed portfolio trading strategy will not be optimal in equilibrium even if the initial heterogeneity in shares is assumed away (e.g., $\theta_i^0 = 1/I$ for all i) whenever $\gamma_i \neq \gamma_h$ for some i, h. On the other hand, if this is the case under the assumptions of Proposition 8, agents are ex-ante identical and consequently there is no asset trading in equilibrium.

Discussion of the Main Result

To simplify the discussion, suppose that $\gamma_i = \overline{\gamma}$ for all i. Hence, at any state (s, K) the degree of heterogeneity across agents is represented by their relative participation in aggregate non-human wealth. Whenever this participation changes with the current state (s, K), the associated equilibrium portfolios will in general adjust accordingly.

To see this more clearly, consider the evolution of individual non-human wealth. A recursive version of (23) in the Appendix implies that for all (s, K, s'):

$$\begin{aligned}\frac{\phi_i(s', K'(s,K))}{\phi_i(s,K)} &= \frac{\overline{\gamma}}{\phi_i(s,K)}\left[V_P(s', K'(s,K)) - V_P(s,K)\Gamma(s,K,s')\right] \\ &\quad + \Gamma(s,K,s') \\ &\quad + \frac{1}{\phi_i(s,K)}\left[V_w(s,K)\Gamma(s,K,s') - V_w(s', K'(s,K))\right]\end{aligned}$$

where $\Gamma(s,K,s') = \frac{z(s,K)}{z(s',K'(s,K))}\left(\frac{\beta\pi(s,s')}{Q(s,K)(s')}\right)^{1/\sigma}$.

This basically says that agent i's stochastic growth of wealth in general depends on his individual level of wealth. This holds except in the case where minimum consumption requirements are 0 for all agents and labor is unproductive. In that particular case, the fraction of non-human aggregate wealth for each agent is kept constant at θ_i^0. At any aggregate state (s,K), individual wealth will be perfectly correlated with aggregate wealth. Thus, individual risk coincides with aggregate risk and consequently there is no trade in equilibrium.

When minimum consumption requirements are zero but there is labor income (e.g. $w(s,K) > 0$ for all (s,K)), it is interesting to notice that each agent $i's$ participation in *aggregate total wealth*, given by $V_F(s,K) + V_w(s,K)I$, is kept constant at $\left(\alpha_i^0\right)^{1/\sigma}$ as time and uncertainty unfold. Equilibrium portfolios must then adjust such that this property holds if $V_F(s,K)$ and $V_w(s,K)$ are not perfectly linearly correlated. In this case, changes in the stock of capital will impact differently on non-human wealth among agents generating changes in individual portfolios. On the other hand, if $V_F(s,K)$ and $V_w(s,K)$ are perfectly correlated, the distribution of consumption is independent of the initial K_0 and the evolution of the aggregate stock of capital will have no impact on the individual composition of wealth. Consequently, the equilibrium level of trading is reduced to zero since all agents share the same kind of risk (e.g. individual risk and aggregate risk coincide again).

An important issue here would be to determine which class of economies displays perfectly linearly correlated levels of wealth. The next result deals with a fundamental particular case where the production function is Cobb-Douglas. Since in this case the value of human wealth is a constant fraction of the value of aggregate output, $V_F(s,K)$ and $V_w(s,K)$ are perfectly correlated if and only if preferences are logarithmic and capital fully depreciates. Only if those two assumptions are satisfied, dividends are also constant fraction of output and thus the value of non-human wealth is perfectly correlated with the value of aggregate output. On the other hand, suppose that capital does not fully depreciate. $V_w(s,K)$ is still a linear function of the value of aggregate output. But it is known that investment is not a constant fraction of output in this case. This implies that dividends are not a linear function of output and thus $V_F(s,K)$ and $V_w(s,K)$ cannot be perfectly correlated. We can summarize this discussion as follows.

Corollary 9 *Suppose that $F(K,L) = K^{\upsilon}L^{1-\upsilon}$ where $\upsilon \in (0,1)$. If $\gamma_i = 0$ for all i, then the fixed equilibrium portfolio property holds if and only if $\delta = \sigma = 1$.*

With unproductive labor but nonzero minimum consumption requirements, the intuition is very similar. In that case, these γ_i's work as "negative endowment" for each agent and thus its value is one of the determinants to explain the evolution of individual wealth relative to the aggregate level. Again, note that the participation of each agent i in aggregate total wealth, now given by $V_F(s,K) - \gamma V_P(s,K)$, is kept constant in this case as well. Thus, equilibrium portfolios will adjust conditional upon (s,K) such that this holds. Under these assumptions, the intertemporal elasticity of substitution is increasing in wealth and consequently the wealthy should hold a relatively higher share of aggregate risk.

4 Conclusions

This paper studies equilibrium portfolios in the traditional one-sector stochastic neoclassical growth model with heterogeneous agents and dynamically complete markets. Preferences are purposely restricted such that Engel curves are affine linear in lifetime human and non-human wealth. Heterogeneity across agents can arise due to differences in initial non-human wealth and minimum consumption requirements. The analysis shows how to explicitly determine equilibrium portfolios as a function of preference parameters, initial endowments and different sources of wealth.

JKS [2003] have seriously questioned the ability of the stationary Lucas tree model to generate nontrivial asset trading under alternative complete market structures and fairly general patterns of heterogeneity across agents. They show that agents choose a fixed equilibrium portfolio which is independent of the state of nature. They conclude that some friction (informational, financial, etc.) must play a significant role in generating nontrivial asset trading in that framework.

In this paper, a crucial aspect independently pointed out by Bossaerts and Zame [2005] and Espino and Hintermaier [2005] has been isolated and studied in more detail. That is, whenever the environment under study generates changing degrees of heterogeneity across agents, the trading strategy of fixed portfolios cannot be optimal in equilibrium.

This paper purposely abstracts from all kind of frictions to study the impact of capital accumulation on the evolution of equilibrium portfolios. We have shown that our environment can generate changing heterogeneity if and only if either minimum consumption requirements are not zero or labor income is not zero and the value of human and non-human wealth are linearly independent.

The theoretical framework analyzed here can be extended to study more general market structures. In particular, the discussion about alternative dynamically complete market structures in Espino and Hintermaier [2005] can be easily adapted to deal with our framework. It is also immediate to extend the analysis to include individual endowments as an additional source of wealth, stochastic minimum consumption requirements, labor-leisure choices and more consumption goods.[10] Furthermore, this setting can also be adapted to deal with accumulation technologies allowing for adjustment costs as in Jermann [1998, 2005] and Boldrin, Christiano and Fisher [2001]. This literature has shown that the introduction of adjustment costs was important to study quantitatively asset returns in production economies.

[10] For the two last extensions mentioned, preferences must be properly restricted (see Townsend [1987]).

Very importantly, we understand that the framework presented here is the natural benchmark to test through quantitative experiments the predictions of the celebrated neoclassical growth model in terms of the evolution of asset holding and stock trading volume. We do not claim that these crucial quantitative aspects will be fully explained by the simple setting presented here. Some other candidates to generate additional trading might be required. However, we do consider important to disentangle the specific contribution of each of those alternative sources and thus the environment described here seems a natural first step.

Appendix

Let $P(s^{t+n}/s^t)$ be the price of one unit of the consumption good delivered at s^{t+n} in terms of the s^t-consumption good (where $P(s^t/s^t) = 1$). We define $V_P(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t)$. Let $M(s^{t+n}/s^t) = \frac{P(s^{t+n}/s^t)}{\beta^{t+n}\pi(s^{t+n})}$ and define

$$V_X(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) \left(M(s^{t+n}/s^t)/\beta^t \pi(s^t)\right)^{\frac{-1}{\sigma}}.$$

The value of human wealth at s^t is given by

$$V_w(s^t) = \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) w(s^{t+n}/s^t),$$

which is the same for all agents.

Proof of Lemma 2. Given our assumptions, we can fully characterize an interior solution for the agent's problem as follows.[11] Let $\lambda_i(s^t)$ be the Lagrange multiplier corresponding to agent i's budget constraint at s^t. Agent i's necessary and sufficient first order conditions imply that:

$$\beta^t \pi(s^t) \left(c_i(s^t) - \gamma_i\right)^{-\sigma} = \lambda_i(s^t), \tag{19}$$

$$\lambda_i(s^t, s') = q(s^t)(s')\lambda_i(s^t), \tag{20}$$

$$p(s^t) = \sum_{s'} q(s^t)(s')[p(s^t, s') + d(s^t, s')]. \tag{21}$$

for all s^t and all s'. Note that (21) coupled with (5) implies that the budget constraint can be rewritten as follows:

$$c_i(s^t) + \sum_{s'} q(s^t)(s')\phi_i(s^t, s') = \phi_i(s^t) + w(s^t), \tag{22}$$

for all s^t. Using (20) and given the definitions of $P(s^{t+n}/s^t)$, $M(s^{t+n}/s^t)$, $V_X(s^t)$, $V_P(s^t)$ and $V_w(s^t)$, note that (19) can be rewritten as follows:

$$c_i(s^{t+n}) - \gamma_i = \lambda_i(s^t)^{\frac{-1}{\sigma}} \left(\frac{P(s^{t+n}/s^t)}{\beta^{t+n}\pi(s^{t+n})}\right)^{\frac{-1}{\sigma}},$$

[11] It is well-known that the necessary and sufficient transversality condition will hold in this setting.

for all s^{t+n}. Using this expression and the necessary transversality condition in (22), we can repeatedly replace non-human wealth to get for each i :

$$\begin{aligned}
&\left(\lambda_i(s^t)\right)^{\frac{-1}{\sigma}} M(s^t)^{\frac{-1}{\sigma}} \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t) \left(\frac{M(s^{t+n}/s^t)}{M(s^t)}\right)^{\frac{-1}{\sigma}} \\
= \quad & \phi_i(s^t) + V_w(s^t) - \gamma_i \sum_{n=0}^{\infty} \sum_{s^{t+n}/s^t} P(s^{t+n}/s^t),
\end{aligned}$$

for all s^t. Solving for $\lambda_i(s^t)$, individual consumption can then be expressed by:

$$c_i(s^t) = \gamma_i \left(1 - b(s^t)\right) + z(s^t) \left(\phi_i(s^t) + V_w(s^t)\right),$$

as in (6), where $b(s^t) = \left[V_X(s^t)\right]^{-1} V_P(s^t)$ and $z(s^t) = \left[V_X(s^t)\right]^{-1}$. To determine $\phi_i(s^t, s')$ (and thus the evolution of individual non-human wealth), observe that (19) and (20) imply:

$$(c_i(s^t, s') - \gamma_i) = \left(\frac{q(s^t)(s')}{\beta\pi(s_t, s')}\right)^{-1/\sigma} (c_i(s^t) - \gamma_i),$$

and thus (6) implies that for all (s^t, s') :

$$\begin{aligned}
\phi_i(s^t, s') \quad = \quad & \gamma_i(V_P(s^t, s') - \Gamma(s^t, s')(V_P(s^t, s')) + \Gamma(s^t, s')\phi_i(s^t) \\
& -(V_w(s^t, s') - \Gamma(s^t, s')V_w(s^t)),
\end{aligned} \tag{23}$$

where $\Gamma(s^t, s') = \frac{z(s^t)}{z(s^t, s')} \left(\frac{\beta\pi(s_t, s')}{q(s^t)(s')}\right)^{1/\sigma}$. ■

Let $\mathbf{C}(S \times X)$ be the set of continuous functions mapping $S \times X$ into the real numbers. For any continuous function $r : S \times X \to \mathbb{R}$, consider the operator T defined by:

$$(TR)(s, K) = r(s, K) + \sum_{s'} Q(s, K)(s') R(s', K'(s, K)).$$

Lemma 10 *Suppose that $r(s, K)$ is one of the following functions:*

(a) $C(s, K)$, (b) $w(s, K)$, (c) $d(s, K)$, (d) a contant, (e) $\frac{C(s', K'(s, K)) - \gamma}{C(s, K) - \gamma}$.

Then, in all these cases there exists a ***unique continuous function*** *$R \in \mathbf{C}(S \times X)$ such that $R(s, K) = (TR)(s, K)$ for all (s, K).*

Proof of Lemma 10. Since the value function V solving (RAPP) is strictly increasing, strictly concave and differentiable in the interior of X and the corresponding policy functions are continuous, we can proceed as in Espino and Hintermaier [2005]. Technical details are available upon request. ■

Proof of Proposition 4. We normalize $\sum_{j=1}^{I} (\alpha_j)^{1/\sigma} = 1$. Let $(C(s,K), K'(s,K))$ be the policy functions solving the problem (RAPP). Compute first the value of aggregate consumption as the solution of the following functional equation:

$$V_C(s,K) = C(s,K) + \sum_{s'} Q(s,K)(s')V_C(s', K'(s,K)). \tag{24}$$

Note that:

$$V_F(s,K) = d(s,K) + \sum_{s'} Q(s,K)(s')V_F(s', K'(s,K)),$$

is independent of α. Since $w(s,K) = sF_2(K,I)$, we have that:

$$V_w(s,K) = w(s,K) + \sum_{s'} Q(s,K)(s')V_w(s', K'(s,K)),$$

is independent of α as well. Since $C(s,K) = d(s,K) + w(s,K)I$, it follows that $V_C(s,K) = V_F(s,K) + V_w(s,K)I$ for all (s,K). Lemma 10 implies that there exist unique continuous functions V_C, V_F and V_w.

The value of individual consumption corresponding to the Pareto optimal allocation α is given by:

$$V_C^i(s,K,\alpha) = c^i(s,K,\alpha) + \sum_{s'} Q(s,K)(s')V_C^i(s', K'(s,K)), \tag{25}$$

and thus individual consumption (2) implies that:

$$\begin{aligned} V_C^i(s,K,\alpha) &= (\alpha_i)^{1/\sigma} V_C(s,K) + V_P(s,K)[\gamma_i - (\alpha_i)^{1/\sigma}\gamma], \\ &= (\alpha_i)^{1/\sigma} [V_F(s,K) + V_w(s,K)I] + V_P(s,K)[\gamma_i - (\alpha_i)^{1/\sigma}\gamma], \end{aligned}$$

solves (25) by definition of V_C and V_P. Also notice that $\phi_i(s,K,\alpha) = V_C^i(s,K,\alpha) - V_w(s,K)$ and consequently it follows that:

$$\phi_i(s,K,\alpha) = (\alpha_i)^{1/\sigma} V_F(s,K) + V_P(s,K)[\gamma_i - (\alpha_i)^{1/\sigma}\gamma] + V_w(s,K)[I(\alpha_i)^{1/\sigma} - 1]. \tag{26}$$

Given some initial state (s_0, K_0) and some initial distribution θ^0, we are ready to compute $\alpha(s_0, K_0, \theta^0)$. Note that by definition:

$$\phi_i(s_0, K_0, \alpha(s_0, K_0, \theta^0)) = \theta_i^0 \left[p(s_0, K_0) + d(s_0, K_0) \right],$$

for all i. Therefore, it follows that:

$$\begin{aligned}
\theta_i^0 \left[p(s_0, K_0) + d(s_0, K_0) \right] &= \theta_i^0 V_F(s_0, K_0), \\
&= \left(\alpha_i(s_0, K_0, \theta^0) \right)^{1/\sigma} V_F(s_0, K_0) \\
&\quad + V_P(s_0, K_0)[\gamma_i - \left(\alpha_i(s_0, K_0, \theta^0) \right)^{1/\sigma} \gamma] \\
&\quad + V_w(s_0, K_0)[I \left(\alpha_i(s_0, K_0, \theta^0) \right)^{1/\sigma} - 1],
\end{aligned}$$

and then,

$$\left(\alpha_i(s_0, K_0, \theta^0) \right)^{1/\sigma} = \frac{\theta_i^0 V_F(s_0, K_0) + V_w(s_0, K_0) - V_P(s_0, K_0)\gamma_i}{V_F(s_0, K_0) + V_w(s_0, K_0) I - V_P(s_0, K_0)\gamma},$$

for $i = 1, ..., (I-1)$. Put $\alpha_I(s_0, K_0, \theta^0) = \left(1 - \sum_{j=1}^{I-1} \left(\alpha_j(s_0, K_0, \theta^0) \right)^{1/\sigma} \right)^{\sigma}$. The assumption that the initial distribution of shares are "large enough" means that:

$$\theta_i^0 V_F(s_0, K_0) + V_w(s_0, K_0) - V_P(s_0, K_0)\gamma_i > 0,$$

for all i. The solutions for both, the planner's problem on one hand, and agents' and the representative firm's problems in a decentralized economy on the other, can be characterized with FOC's. Thus, it can be checked that the candidate recursive allocation and the price system proposed can be decentralized as a RCE. [12] ■

Proof of Proposition 5. To see this, supose that $\delta = 0$ and $F(K, L) = F(L)$ for all (K, L). Thus, $y(s) = sF(I) + K_0$ represents the fruit delivered by this tree at state s and $w(s) = sF'(I)$ is the wage per unit of labor. Since the Second Welfare Theorem holds, any Pareto optimal allocation (parametrized by α), $c_i(\alpha) \in \mathbb{R}^S$, can be decentralized as a RCE with transfers. Equation (14) reduces to:

$$\phi_i(s, \alpha) = c_i(s, \alpha) - w(s) + \sum_{s'} Q(s)(s')\phi_i(s', \alpha).$$

[12] See Espino and Hintermaier [2005] for technical details in a similar environment.

for $s = 1, ..., S$. As pointed out by JKS [2003], this is a S-dimensional system with a unique solution continuous in α. Continuity with respect to α implies that there exists α^0 such that $\phi_i(s_0, \alpha^0) = \theta_i^0 \left[p(s_0) + d(s_0)\right]$ for all i. Consequently, the resulting equilibrium portfolio in this economy is independent of (s, Φ) and given by:

$$a_i'(s, \Phi)(s') = \phi_i(s', \alpha^0) - \theta_i^0 \left[p(s') + d(s')\right],$$

where $\Phi_i = \phi_i(s, \alpha^0)$ for each i. ∎

Proof of Proposition 6. Consider $\alpha^0 = \alpha^0(s_0, K_0, \theta_i^0)$ given by (16) and note that equilibrium portfolios will be determined by:

$$\begin{aligned} A_i(s, K, \alpha^0) &= [\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0] V_F(s, K) + [\gamma_i - \left(\alpha_i^0\right)^{1/\sigma} \gamma] V_P(s, K) \\ &\quad + [I \left(\alpha_i^0\right)^{1/\sigma} - 1] V_w(s, K). \end{aligned}$$

If we assume that $\gamma_i = V_w(s, K) = 0$ for all i and for all (s, K), then it follows from (16) that $\left(\alpha_i^0\right)^{1/\sigma} = \theta_i^0$. Therefore, (17) implies that $A_i(s, K, \alpha^0) = 0$ for all (s, K) and for all i. ∎

Proof of Proposition 7. Suppose that $\gamma_i > 0$ for some agent i and $V_w(s, K) = 0$ for all (s, K). It follows from (17) that:

$$A_i(s, K, \alpha^0) = [\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0] V_F(s, K) + [\gamma_i - \left(\alpha_i^0\right)^{1/\sigma} \gamma] V_P(s, K).$$

The fixed equilibrium portfolio property means that $A_i(s, K, \alpha^0) = A_i(s, \alpha^0)$ for all i and for all (s, K). In particular, it implies that $A_i(s_0, K, \alpha^0) = 0$ for all K by definition of α^0. Then,

$$[\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0] V_F(s_0, K) = [\left(\alpha_i^0\right)^{1/\sigma} \gamma - \gamma_i] V_P(s_0, K). \tag{27}$$

for all K and thus it follows from (16) that α^0 is independent of K (e.g. $\alpha(s_0, \theta^0) = \alpha(s_0, K, \theta^0)$ for all K). Note that (27) implies that there exists $x > 0$ such that $V_F(s_0, K) = x V_P(s_0, K)$ for all K. In this case the dividend policy given by:

$$d(s^t) = s_t f(K(s^{t-1})) + (1 - \delta) K(s^{t-1}) - K(s^t) = x \text{ for all } s^t,$$

is the unique solution for (FP), given the definitions of $V_F(s_0, K_0)$ and $V_P(s_0, K_0)$. This would imply that $d(s^t) = C(s^t) = x$ for all s^t solves the planner's problem (APP). However, this would violate well-known standard properties of the one-sector neoclassical growth model. ■

Proof of Proposition 8. Suppose that $\gamma_i = 0$ for all i and $V_w(s, K) > 0$ for all (s, K). It follows from (17) that:

$$A_i(s, K, \alpha^0) = [\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0]V_F(s, K) + [I\left(\alpha_i^0\right)^{1/\sigma} - 1]V_w(s, K).$$

Suppose that the fixed equilibrium portfolio property holds where $A_i(s, K, \alpha^0) = A_i(s, \alpha^0)$ for all i and for all (s, K). Fix some i and note that this implies that for all (s, K) :

$$V_F(s, K) = \frac{A_i(s, \alpha^0)}{[\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0]} - \frac{[I\left(\alpha_i^0\right)^{1/\sigma} - 1]}{[\left(\alpha_i^0\right)^{1/\sigma} - \theta_i^0]}V_w(s, K)$$

and thus $V_F(s, K)$ and $V_w(s, K)$ need to be perfectly linearly correlated. Also notice that $A_i(s_0, \alpha^0) = 0$ for all i implies that α^0 is again independent of K_0. ■

Proof of Corollary 9. The fixed equilibrium portfolio property holds if and only if $A_i(s, K, \alpha^0) = A_i(s, \alpha^0)$ for all i and for all (s, K), where $A_i(s_0, \alpha^0) = 0$ for all i. We have shown that this implies that $V_F(s_0, K) = xV_w(s_0, K)$ for some $x > 0$ whenever $\gamma_i = 0$ for all i. If $F(K, L) = K^{\upsilon}L^{1-\upsilon}$, then $V_w(s, K) = \frac{(1-\upsilon)}{I}sf(K) + \sum_{s'} Q(s, K)(s')V_w(s', K'(s, K))$. Given (s_0, K_0), the capital accumulation path given by:

$$K(s^t) = (1 - (1-\upsilon)(1 + \frac{x}{I}))s_t f(K(s^{t-1})) + (1-\delta)K(s^{t-1}) \text{ for all } s^t,$$

attains $V_F(s_0, K_0)$ given its definition in (FP) above. Since the solution is unique, this would imply that $C(s^t) = (1-\upsilon)(1+\frac{x}{I})s_t f(K(s^{t-1}))$ for all s^t solves the planner's problem (APP). It is known that this last equality holds if and only if $\delta = \sigma = 1$ (see Benhabib and Rustichini [1994]). ■

References

[1] ALVAREZ, F. AND STOKEY, N. [1998]: "Dynamic Programming with Homogenous Functions", *Journal of Economic Theory*, 82, 167-189.

[2] BENHABIB, J. AND RUSTICHINI, A. [1994]: "A Note on a New Class of Solutions to Dynamic Programming Problems Arising in Economic Growth", *Journal of Economic Dynamic and Control*, 18, 807-813.

[3] BOLDRIN, M., CHRISTIANO, L. AND FISHER, J. [2001]: "Habit Persistence, Asset Returns, and the Business Cycle", *American Economic Review*, 91 (1), 149-166.

[4] BOSSAERTS, P. AND W. ZAME [2005]: "Asset Trading Volume in Infinite-Horizon Economies with Dynamically Complete Markets and Heterogeneous Agents: Comment", Working Paper, California Institute of Technology.

[5] BROCK, W. A. [1982]: "Asset Prices in a Production Economy", in *The Economics of Information and Uncertainty*, ed. by J. J. McCall, pp. 1–46. The University of Chicago Press, Chicago.

[6] BROCK, W. AND MIRMAN [1972]: "Optimal Economic Growth and Uncertainty: the Discounted Case", *Journal of Economic Theory*, 4, 479-513.

[7] CHATTERJEE, S. [1994]: "Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model", *Journal of Public Economics,* 54, 97-119.

[8] ESPINO, E. AND HINTERMAIER, T. [2005]: "Asset Trading Volume in a Production Economy", Working Paper, IHS.

[9] JERMANN, U. [1998]: "Asset Pricing in Production Economies", *Journal of Monetary Economics,* vol. 41, 257–75.

[10] JERMANN, U. [2005]: "The Equity Premium Implied by Production", Working paper, Wharton School.

[11] JUDD, K., KUBLER, F. AND SCHMEDDERS, K. [2003]: "Asset Trading Volume in Infinite-Horizon Economies with Dynamically Complete Markets and Heterogeneous Agents", *Journal of Finance*, 58, 2203-2217.

[12] LE VAN, C. AND MORHAIM, L. [2002]: "Optimal Growth Models with Bounded or Unbounded Returns: A Unifying Approach", *Journal of Economic Theory*, 105, 158–187.

[13] LJUNGQVIST, L. AND SARGENT, T. [2004]: *Recursive Macroeconomic Theory. Second Edition.* Cambridge, Mass.: The MIT Press.

[14] LUCAS, R. E. JR. [1978]: "Asset Prices in an Exchange Economy", *Econometrica*, 46, 1429-1445.

[15] MEHRA, R. AND PRESCOTT, E.C. [1985]: "The Equity Premium: A Puzzle", *Journal of Monetary Economics*, 15, 145–161.

[16] TOWNSEND, R. [1987]: "Arrow-Debreu Programs as Microfoundations for Macroeconomics", in *Advances in Economic Theory*: The Fifth World Congress, Truman F. Bewley, ed., Cambridge University Press, 1987.

[17] OBIOLS-HOMS, F. AND URRUTIA, C. [2005]: "Transitional Dynamics and the Distribution of Assets", *Economic Theory*, 25 (2), 381-400.

[18] NEGISHI, T. [1960]: "Welfare Economics and the Existence of an Equilibrium for a Competitive Economy", *Metroeconomica,* 7, 92-97.

[19] STOKEY, N., LUCAS, R. E. JR. AND PRESCOTT, E.C. [1989]: *Recursive Methods in Economic Dynamics*, Cambridge, MA: Harvard University Press.