Universidad de
SanAndrés

Departamento de Economía
Ciclo de Seminarios

A Uni...ed Framework for Monetary Theory and Political Analysis

Ricardo Lagos (New York University)

Martes 28 de mayo de 2002
Aula Jorge Born

Departamento de Economía

Director
Mariano Tommasi
Ph.D. in Economics, University of Chicago

Coordinador del Ciclo de Seminarios
Martín Gonzales Eiras
Ph.D. in Economics, MIT

En Internet:
http://www.udesa.edu.ar/deptodeeconomia/dtoeconomiappal.htm

UdeSA

A Uni...ed Framework for Monetary Theory and Policy Analysis[¤]

Ricardo Lagos
New York University

Randall Wright
University of Pennsylvania

QUAERERE VERUM

February 27, 2002

Abstract

Tractable versions of search-theoretic models of money often rely on assumptions that make some policy experiments, especially changes in the money supply, di¢cult to interpret, at best. Perhaps for this reason, economists interested in monetary policy, typically resort to reduced-form models like those that involve cash-in-advance constraints, money in the utility function, and so on. In this paper we present a framework that attempts to bridge the gap between the pure theory of money and the relatively applied branch of the literature that emphasizes policy. The model is based explicitly on the frictions that make money essential, as in the search-based monetary literature, but at the same time allows for fairly general monetary policies, as do reduced-form models. In this sense it helps to integrate micro and macro strands in monetary economics.

Universidad de San Andrés

[¤]Financial support from the CV Starr Center for Applied Economics at NYU, STICERD and the NSF is gratefully acknowledged. First version: May 2000.

BIBLIOTECA UNIVERSIDAD DE SAN ANDRES MAX VON BUCH

Sem. Eco. 0217

SN 29981

1 Introduction

This paper is an attempt to provide a unification, or at least to develop some common ground, between microeconomic and macroeconomic models of monetary exchange. Why is this desirable? First, existing macro models are all to some extent reduced-form models. By this we mean they make assumptions, such as putting money in the utility function or imposing cash-in-advance constraints, that are presumably meant to stand in for some role for money that is not made explicit but ought to be – say, that it helps overcome spatial, temporal, or informational frictions. Second, attempts to provide micro foundations for monetary economics using search theory, with explicit descriptions of specialization, the pattern of meetings, the information structure, and so on, typically need severe restrictions for tractability. For example, there are typically extreme restrictions on how much money agents can hold, which makes the analyses of some policy issues difficult at best.[1]

We have several goals. We want a framework that, like existing macro models, allows one to analyze standard issues in monetary economics in both a qualitative and quantitative fashion; an example is to determine the welfare cost of inflation. At the same time we want a model where the role for money is explicit, which allows us to address some issues that can be studied more

[1] In terms of the literature, the reduced form approach is far too vast to go into here, but examples include Cooley and Hansen (1989) and Christiano, Eichenbaum and Evans (1997); see Walsh (1998) for other references. We will go into much more detail below on the search-based literature, but examples include Kiyotaki and Wright (1991, 1993), Shi (1995), Trejos and Wright (1995) and Kocherlakota (1998).

naturally with search-based than reduced-form models; examples include, to ask exactly what frictions make the use of money an equilibrium or an efficient arrangement, and to show how different regimes (say, commodity versus fiat money) lead to different outcomes. Finally, we want the framework to be tractable and capable of delivering clean analytic results, but at the same time we want it to be close enough to the actual economy that it can be relatively easily and realistically calibrated.

There are of course previous attempts to provide models with micro foundations but without the severe restrictions on money holdings. An example is Molico (1999), who allows agents to hold any $m \in \mathbb{R}_+$. This greatly extends the set of issues that could be studied using previous search-based monetary models, but Molico's framework is extremely complicated – not many results are available, except those found by computation, and even numerically the model is quite difficult to analyze. One of the main problems is that the distribution of money holdings across agents, $F(m)$, is endogenous. There is also the approach pioneered by Shi (1997) and extended since by several people, which gets around this problem by making some creative assumptions to render $F(m)$ degenerate (additional discussion and references are provided below).

In our model, $F(m)$ will also be degenerate, although the economic environment and the technical details by which we get this will differ significantly from Shi's. We will have much more to say about the similarities and differences between our framework and various alternatives later. Here we simply want to emphasize that at the end of the day some of the results will look

Universidad de SanAndrés

similar to these previous attempts to integrate micro and macro monetary models, and indeed some of the results will look much like those one finds in reduced form models. This is as it should be: those models are meant to be descriptive of what we see in actual economies, and to the extent that it is successful, any good model should end up on some dimensions making similar predictions. At the same time it seems desirable to have micro foundations, and making these explicit leads to much economic insight, as we shall discuss below.

The rest of the paper is organized as follows. Section 2 presents the basic model. It is divided into four Subsections, where we introduce the environment, define equilibrium, give the main economic results, and compare our model to the related literature. Section 3 presents extensions. It is divided also into four Subsections, where we discuss monetary policy and welfare, analyze dynamics, introduce real shocks, and introduce monetary shocks. Section 4 summarizes the results and discusses some possible future research topics. A few technical results on bargaining and dynamic programing are relegated to the Appendix.

2 The Basic Model

2.1 Environment

Time is discrete. There is a $[0,1]$ continuum of agents who live forever and have discount factor $\beta \in (0,1)$. In the interest of integrating standard macroeconomic and search-theoretic models, we assume that there are two types of commodities in the economy: *general* and *special* goods. As in standard macro models, all agents consume and produce the general good. The utility of consuming X units of this goods is $U(X)$ and the disutility of producing Y units is $C(Y)$. One interpretation is that agents literally produce the good themselves, but it is equivalent to saying they supply labor h at disutility $C(h)$, and firms convert this labor (and capital) into Y via a standard production function. For now we adopt the former interpretation. What is important is that either U or C is linear. Here we assume that $C(Y) = Y$, and that U is C^2 with $U' > 0$ and $U'' \leq 0$. We assume U is either unbounded, or if it has a bound it satisfies a condition given in Lemma 4, and that $U'(X^*) = 1$ for some $X^* \in (0, \infty)$.

In contrast to general goods, each agent produces a subset and consumes a subset of special commodities, as in a typical search model. Specialization is modeled here as follows. Given two agents i and j drawn at random, there are four possible events. The probability that both consume something the other can produce (a double coincidence) is denoted δ. The probability that i consumes something j produces but not vice-versa (a single coincidence) is denoted σ. Symmetrically, the probability that j consumes something i produces but not vice-versa is also σ. Finally, the probability that neither wants anything the other can produce is $1 - 2\sigma - \delta$, where $\sigma \leq (1 - \delta)/2$ is assumed.[2] In any single coincidence meeting, if i wants what j produces, we

[2] This notation captures several explicit specifications for specailization in the literature as special cases. For example, in Kiyotaki and Wright (1989) or Aiyagari and Wallace (1991) there are N goods and N types, where type n produces good n and consumes good $n + 1$ (mod N). If $N > 2$ we have $\sigma = 1/N$ and $\delta = 0$, while if $N = 2$ we have $\delta = 1/2$ and $\sigma = 0$. In Kiyotaki and Wright (1993), the event that i consumes what j

call i the *buyer* and j the *seller*.

Let $u(q)$ be the utility of consumption and $c(q)$ the disutility production of any special good, where u and c are C^n with $n > 2$. We assume $u(0) = c(0) = 0$, $u'(q) > 0$, $c'(q) > 0$, $u''(q) < 0$, $c''(q) \geq 0$, and $u(\bar{q}) = c(\bar{q})$ for some $\bar{q} > 0$. We use q^* to denote the efficient quantity of special good production, which solves $u'(q^*) = c'(q^*)$; q^* is what all agents would agree to ex ante if they had some way of committing to or enforcing the agreement. Note that we can always normalize $c(q) = q$, without loss in generality, as long as we rescale $u(q)$; this merely amounts to measuring output in utils rather than physical units. At one point in the analysis we use a condition on the third derivative, $u''' \leq (u'')^2/u'$. A simple way to state this condition is to say that marginal utility is log concave, since it is obviously equivalent to assuming that the second derivative of $\log u'(q)$ is negative.[3]

Both general and special goods are nonstorable, but there is another object called *money* that can be stored. Money, like goods, is perfectly divisible and agents can hold any quantity $m \geq 0$. Money has no intrinsic value, but could potentially be used to trade for consumption goods. We emphasize that it is not necessary to use money in trade, and it is generally possible, for example, to exchange one special good directly for another. However, one cannot trade special for general goods due to the following assumption: in each period there are two sub-periods, day and night, and special goods

produces is independent of the event that j consumes what i produces, and each occurs with probability x. Then $\delta = x^2$ and $\sigma = x(1-x)$.

[3]Dating back to Burdett (1981), the log concavity of certain objects has proved useful in many applications in search theory.

are only produced during the day while general goods are only produced at night. Given consumption goods are nonstorable, the only feasible trades during the day are barter in special goods or the exchange of special goods for money, and the only feasible trades at night are barter in general goods or the exchange of general goods for money. See Figure 1.[4]

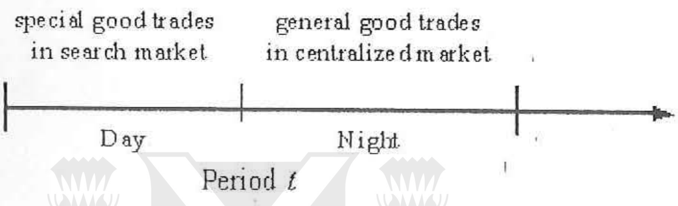

Figure 1: Timing

During the day agents participate in a bilateral matching process, as in standard search theory. In this decentralized market there is a probability α of a meeting each period, each meeting is a random draw from the population, and the terms of trade are determined by bargaining. At night there is a frictionless centralized market where one dollar buys ϕ units of general goods – i.e., $p_g = 1/\phi$ is the nominal price, and agents take it parametrically. All trade in the decentralized market must be quid pro quo, either goods for goods or goods for money; there is no credit, because the matching

QUAERERE VERUM
Universidad de
SanAndrés

[4]It is equivalent for everything we do to assume the opposite scenario, where general goods are produced in the first subperiod and special good during the second, or to have general good produced in even periods and and special in odd periods. The distinction is that agents discount at rate $\beta < 1$ between periods but not between subperiods. By having two discount factors, say β_1 between day and night and β_2 between night and day, one could actually nest the different alternatives in one model, but at a cost in terms of notation.

process is anonymous and hence there is no punishment for reneging on debt (Kocherlakota [1998]; Wallace [2000, 2001]). We could allow intertemporal trade in general goods, but in equilibrium it will not happen, at least in the basic model with no intrinsic heterogeneity (we can price a bond, but it will not trade since we cannot find one agent who wants to save and another to borrow at the same interest rate).

2.2 Equilibrium

In this subsection we build gradually towards the definition of equilibrium. We begin by describing the value functions, taking as given the terms of trade and the distribution of money. In general, the state variable for an individual includes his own money holdings m and a vector of aggregate states s. At this point we let $s = (\phi, F)$, where ϕ is the value of money in the centralized market and F is the distribution of money holdings in the decentralized market – i.e., $F(\tilde{m})$ is the measure of agents in this market holding $m \leq \tilde{m}$. Necessarily F satisfies $\int m dF(m) = M$ at every date, where M is the total money stock that is fixed (until the next section). The agent takes as given the law of motion $s_{+1} = \Upsilon(s)$, but it will be determined in equilibrium. If fact, in equilibrium, F_{+1} will degenerate at $m = M$ for any F, and ϕ_{+1} will be given by a function $\Phi(\phi)$ to be determined.

Let $V(m, s)$ be the value function for an agent with m dollars in the morning when he enters the decentralized market, and $W(m, s)$ the value function in the afternoon when he enters the centralized market, given s. Let $q(m, \tilde{m}, s)$ and $d(m, \tilde{m}, s)$ be the quantity of goods and dollars that change hands in a single coincidence meeting between a buyer with m dollars and a seller with $\tilde{m}$ dollars, given s. Let $B(m,\tilde{m},s)$ be the payoff from a barter trade for an agent with m dollars who meets an agent with $\tilde{m}$ and there is a double coincidence of wants, given s. Then we have the Bellman equation[5]

$$\begin{aligned} V(m,s) &= \alpha\sigma \int \{u[q(m,\tilde{m},s)] + W[m - d(m,\tilde{m},s)]\} dF(\tilde{m}) \\ &\quad + \alpha\sigma \int \{-c[q(\tilde{m},m,s)] + W[m + d(\tilde{m},m,s)]\} dF(\tilde{m}) \quad (1) \\ &\quad + \alpha\delta \int B(m,\tilde{m},s) dF(\tilde{m}) + (1 - 2\alpha\sigma - \alpha\delta) W(m,s). \end{aligned}$$

The value of entering the centralized market with m dollars is

$$\begin{aligned} W(m,s) &= \max_{X,Y,m_{+1}} \{U(X) - Y + \beta V(m_{+1}, s_{+1})\} \\ s.t.\ X &= Y + \phi m - \phi m_{+1} \end{aligned}$$

where X is consumption and Y production of general goods, and m_{+1} is money taken out of this market and into the decentralized market next period. Assume for the moment that the constraint $Y \geq 0$ is not binding; then we have

$$W(m,s) = U(X^*) - X^* + \phi m + \max_{m_{+1}} \{-\phi m_{+1} + \beta V(m_{+1}, s_{+1})\} \quad (2)$$

where $U'(X^*) = 1$. This immediately implies that m_{+1} does not depend on

[5]The first term in (1) is the expected payoff from a single coincidence meeting where you buy $q(m,\tilde{m},s)$ and then go to the centralized market with $m - d(m,\tilde{m},s)$ dollars. The second term is the expected payoff from a single coincidence meeting where you sell $q(\tilde{m},m,s)$ and go to the centralized market with $m + d(\tilde{m},m,s)$ dollars (notice the roles of m and $\tilde{m}$ are reversed in the first two terms). The third term is the expected payoff from barter, and the final term is the expected payoff from going to the centralized market without having traded in the decentralized market.

QUAERERE VERUM Universidad de SanAndrés

m, and that W is linear (i.e. affine) in m,

$$W(m, s) = W(0, s) + \phi m. \tag{3}$$

It is not hard to provide assumptions to guarantee $Y \geq 0$ is not binding, and so we will use these results in what follows.[6]

We now consider the terms of trade in the decentralized market, which are determined by bargaining. There are two bargaining situations to consider: single coincidence and double coincidence meetings. In the case of a double coincidence we adopt the symmetric Nash bargaining solution with the threat point of an agent given by his continuation value $W(m, s)$. Lemma 1 in the Appendix shows that, regardless of the money holdings of the two agents, this implies that in any double coincidence meeting the agents give each other the efficient quantity q^* and no money changes hands. If we let $b = u(q^*) - c(q^*)$, this means

$$B(m, \tilde{m}, s) = b + W(m, s). \tag{4}$$

Now consider bargaining in a single coincidence meeting when the buyer has m and the seller $\tilde{m}$ dollars. In this case we use the generalized Nash solution where the buyer has bargaining power θ and threat points are given

[6] A trivial way to get $Y \geq 0$ to not bind is to assume $U = C$. More generally, one can proceed as follows. First notice $Y = Y(m) = X^* + \phi(m_{+1} - m) \geq X^* - \phi m$. We show below that in any equilibrium $\phi M \leq u(q^*)$. Hence, if we assume $X^* > u(q^*)$ we know $Y(M) > 0$. Then, at any point in time, as long as the distribution $F(m)$ is not too disperse we have $Y(m) > 0$ for all m; a sufficient condition for dispersion is $m \leq MX^*/u(q^*)$ with probability 1. Moreover, under conditions given below, $Y(m) > 0$ implies $m_{+1} = M$ for all agents, so the distribution will collapse to a point and then $Y(m) = Y(M) > 0$ for all future dates.

QUAERERE VERUM
Universidad de
San Andrés

by continuation values. That is, (q, d) maximizes

$$[u(q) + W(m-d, s) - W(m, s)]^{\theta} [-c(q) + W(\tilde{m} + d, s) - W(\tilde{m}, s)]^{1-\theta} \tag{5}$$

subject to $d \leq m$. By virtue of (3), this simplifies nicely to

$$\max_{q,d} [u(q) - \phi d]^{\theta} [-c(q) + \phi d]^{1-\theta} \tag{6}$$

subject to $d \leq m$; notice in particular that the continuation values have vanished.[7] Also note that there are implicitly two side conditions, $u(q) \geq \phi d$ and $c(q) \leq \phi d$, but they will not bind and so we can ignore them, unless $\theta = 0$ or 1. They do however immediately imply $u(q) \geq c(q)$, or $q \in [0, \bar{q}]$.

The solution (q, d) to (6) does not depend on $\tilde{m}$, and depends on m only if the constraint $d \leq m$ binds. Also, it depends on s only through ϕ, and indeed only through real balances $z = \phi m$. We abuse notation slightly and write $q(m, \tilde{m}, s) = q(m)$ and $d(m, \tilde{m}, s) = d(m)$ in what follows (the dependence on ϕ is implicit). Lemma 2 in the Appendix shows that

$$q(m) = \begin{cases} \widehat{q}(m) & \text{if } m < m^* \\ q^* & \text{if } m \geq m^* \end{cases} \text{ and } d(m) = \begin{cases} m & \text{if } m < m^* \\ m^* & \text{if } m \geq m^* \end{cases} \tag{7}$$

where

$$m^* = \frac{\theta c(q^*) + (1-\theta) u(q^*)}{\phi}, \tag{8}$$

and $\widehat{q}(m)$ solves the first order condition from (6), which for future reference we write as

$$\phi m = \frac{\theta c(q) u'(q) + (1-\theta) u(q) c'(q)}{\theta u'(q) + (1-\theta) c'(q)}. \tag{9}$$

Universidad de San Andrés

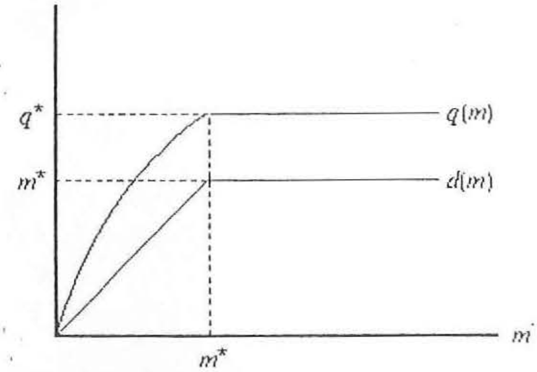

Figure 2: Single-Coincidence Bargaining Solution

The solution is shown in Figure 2. Since u and c are C^n, the implicit function theorem implies $\widehat{q}(m)$ is C^{n-1} and

$$\widehat{q}'(m) = \frac{\phi[\theta u' + (1-\theta)c']}{u'c' - \theta(\phi m - c)u'' + (1-\theta)(u - \phi m)c''}$$

for all $m < m^*$. Inserting ϕm from (9), we have

$$\widehat{q}'(m) = \frac{\phi[\theta u' + (1-\theta)c']^2}{u'c'[\theta u' + (1-\theta)c'] + \theta(1-\theta)(u-c)(u'c'' - c'u'')}. \tag{10}$$

Hence, $\widehat{q}(m)$ is strictly increasing for $m < m^*$ and $\lim_{m \to m^*} \widehat{q}(m) = q^*$, which means $\widehat{q}(m) < q^*$ for all $m < m^*$. Notice $q(m)$ is not differentiable at m^* (the left derivative is strictly positive while the right derivative is 0). Also notice $\widehat{q}(m)$ need not be concave or convex, as this depends on u'''.

We can now insert $W(m)$ together with the bargaining outcomes (4) and

[7] This is nice because it means the Nash solution can be interpreted as the outcome of an explicit strategic bargaining game even if s is not constant over time, something that is *not* true generally in dynamic models (Coles and Wright [1998]).

Universidad de San Andrés

(7) into (1) and rewrite the Bellman equation as

$$V(m, s) = \max_{m_{+1}} \{v(m, s) + \phi m - \phi m_{+1} + \beta V(m_{+1}, s_{+1})\} \tag{11}$$

where

$$v(m, s) = v_0(s) + \alpha\sigma \{u[q(m)] - \phi d(m)\} \tag{12}$$

and $v_0(s) = \alpha\sigma \int \{\phi d(\tilde{m}) - c[q(\tilde{m})]\} dF(\tilde{m}) + \alpha\delta b + U(X^*) - X^*$ is independent of m. Hence, V depends on m separably through a linear term ϕm and a bounded and continuous function $v(m, s)$.[8] This allows us to establish that there exists a unique $V(m, s)$ in the relevant space of functions satisfying (11), even though $V(m, s)$ is unbounded because of the linear term ϕm.

Here we outline the argument for the case where s is constant – which does nothing to overcome the problem of unboundedness, but does simplify the presentation – and relegate the more general case to Lemma 5 in the Appendix. Since s is constant, write $V(m, s) = \hat{V}(m)$. Then consider the space of functions $\hat{V} : \mathbb{R}_+ \to \mathbb{R}$ that can be written $\hat{V}(m) = \hat{v}(m) + \phi m$ for some bounded and continuous function $\hat{v}(m)$. For any two functions in this space $\hat{V}_1(m) = \hat{v}_1(m) + \phi m$ and $\hat{V}_2(m) = \hat{v}_2(m) + \phi m$, we can define $\left\|\hat{V}_1 - \hat{V}_2\right\| = \sup_{m \in \mathbb{R}_+} |\hat{v}_1(m) - \hat{v}_2(m)|$, and this constitutes a complete metric space. One can show the right hand side of (11) defines a contraction

[8] That $v(m, s)$ is bounded and continuous in m follows from the bargaining solution. Later we will see that F is degenerate and $\phi_{+1} = \Phi(\phi)$. Also, we show ϕ is bounded in any equilibrium in Lemma 4 in the Appendix. Note that the trick of putting the current price ϕ in the state vector allows us to capture nonstationary equilibria while still using recursive methods; this was previously used by Duffie et al. (1994) in an overlapping generations model.

mapping $T\hat{V}$. Hence, by the contraction mapping theorem there exists a unique solution to $\hat{V} = T\hat{V}$ in the relevant space.[9]

Given it exists, $V(m,s)$ is C^{n-1} except at $m = m^*$ because $q(m)$ and $d(m)$ are. For $m > m^*$, $V_1(m,s) = \phi$; for $m < m^*$,

$$V_1(m,s) = \alpha\sigma\phi e(q) + (1 - \alpha\sigma)\phi \tag{13}$$

where $e(q) = u'(q)\hat{q}'(m)/\phi$ is the gain from having an additional unit of real balances when bargaining. From (10),

$$e(q) = \frac{[\theta u' + (1-\theta)c']^2 u'}{u'c'[\theta u' + (1-\theta)c'] + \theta(1-\theta)(u-c)(u'c'' - c'u'')}. \tag{14}$$

For future reference note that as $m \to m^*$ from below, $q \to q^*$, and therefore

$$V_1(m,s) \to \frac{\alpha\sigma\phi}{1 + \theta(1-\theta)(u-c)(c''-u'')(u')^{-2}} + (1-\alpha\sigma)\phi < \phi. \tag{15}$$

Thus, the slope of $V(m,s)$ jumps discretely as m crosses m^*.

The next thing to do is to check the concavity of V for $m < m^*$. To reduce notation, at this point, we normalize $c(q) = q$ with no loss of generality, as discussed above. This reduces the algebra required to show that V'' takes the same sign as $\Gamma + (1-\theta)[u'u''' - (u'')^2]$ for all $m < m^*$, where $\Gamma < 0$. From this result it is not possible to sign V'' in general, due to the presence of u''', but it does give us some sufficient conditions for $V'' < 0$. One such condition is $\theta \approx 1$. Another is $u'u''' \leq (u'')^2$, which follows from the assumption that marginal utility is log-concave (given the normalization for c). Hence, we

[9] Operationally, the contraction generates the function $\hat{v}(m)$ and then we simply set $\hat{V}(m) = \hat{v}(m) + \phi m$. Note that this is not the same method for dealing with unbounded returns discussed in Alvarez and Stokey (1998).

have simple sufficient conditions to guarantee that V is strictly concave for all $m < m^*$, given any distribution F and any $\phi > 0$.[10]

To summarize the discussion to this point, we first described the value function in the decentralized market, $V(m, s)$, in terms of $W(m, s)$ and the terms of trade. We then derived some properties of the value function in the centralized market, including $W(m, s) = W(0, s) + \phi m$. This made it relatively easy to solve the bargaining problem for $q(m)$ and $d(m)$. This allowed us to simplify the Bellman equation considerably, to establish the existence of a unique solution, and to give several properties of the value function, including differentiability and strict concavity for $m < m^*$. Although we only sketched some parts of the argument in the text, the details are given in the Appendix, where it is established that the above results hold even when ϕ varies over time according to a continuous function $\phi_{+1} = \Phi(\phi)$ as long as we include ϕ as a state variable.[11]

We now return to the problem of an agent deciding how much cash to take out of the centralized market: $\max_{m_{+1}} \{-\phi m_{+1} + \beta V(m_{+1}, s_{+1})\}$. We

[10]To understand the concavity argument, notice that for $m < m^*$, $V'' = (q')^2 u'' + u' q''$. The first term is negative but the second takes the sign of q'', which could be positive. Intuitively, $q'' > 0$ means that having more money gets you a lot better deal in bargaining. The assumption $\theta = 1$ implies $q(m) = \phi m$, given $c(q) = q$, so $V'' < 0$. If $\theta < 1$ then $q(m)$ is nonlinear, and we need a condition to bound how nonlinear it can be, which is log concavity. When u is not log concave and $\theta < 1$, we constructed examples where q is not concave, but we could not construct an example where V was not concave.

[11]The discussion in the text is circular in the following sense: we will argue below that the properties of V imply some nice results, like F degenerate. But we used F degenerate (or, at least it F constant over time and well-behaved) to prove V exists. This is rectified in the Appendix, where we argue from first principles (i.e., without using the value function) that F is degenerate for all $t > 0$, show how to construct Φ where $\phi_{+1} = \Phi(\phi)$, and prove that ϕ is bounded. These results allow us to establish that there exists a unique solution V to the Bellman equation.

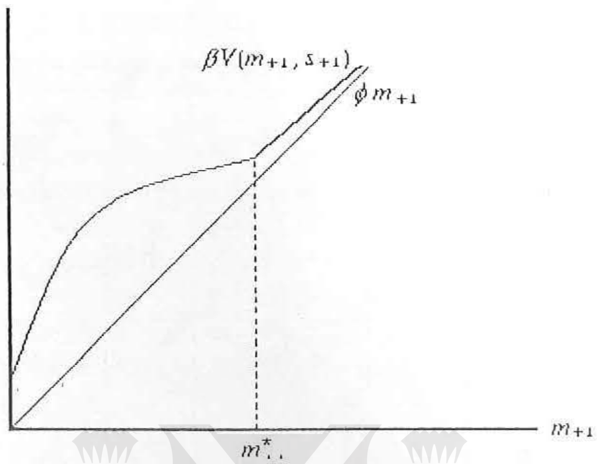

Figure 3: Value Function

claim this problem has no solution if $\phi < \beta\phi_{+1}$. To establish this, simply note that the derivative of $-\phi m + \beta V(m_{+1}, s_{+1})$ is $-\phi + \beta\phi_{+1}$ for all $m_{+1} > m^*_{+1}$. Hence, in any equilibrium we must have $\beta\phi_{+1} \leq \phi$ or else payoffs are unbounded in the control variable m_{+1}. Notice that $\beta\phi_{+1} \leq \phi$ implies $-\phi m_{+1} + \beta V(m_{+1}, s_{+1})$ is nonincreasing for $m_{+1} > m^*_{+1}$ in any equilibrium. But recall from (15) that the slope of $V(m_{+1}, s_{+1})$ just to the left of m^*_{+1} is strictly less than the slope to the right of m^*_{+1}, as shown in Figure 3. From this it is obvious that any solution to this problem m_{+1} must be strictly less than m^*_{+1}.

Given that V is strictly concave over the relevant range, there exists a unique solution and it satisfies

$$\beta V_1(m_{+1}, s_{+1}) - \phi \leq 0, \quad = 0 \text{ if } m_{+1} > 0. \tag{16}$$

The key result is that all agents pick the same m_{+1}. Therefore, at least after the initial date $t = 0$ where we might start from an arbitrary distribution, at all future dates $F(\tilde{m})$ must be degenerate at $\tilde{m} = M$. This means any monetary equilibrium is characterized by a degenerate distribution F for every $t > 0$, and a solution to (16) at equality with $m_{+1} = M$. This of course assumes the condition $Y \geq 0$ does not bind in the general goods market, which is true under the conditions discussed above.[12]

To be more precise, since we know $M = m < m^*$ in any equilibrium, we can now set $d = m$ and $q = \hat{q}(m)$. Recall that $m^* = z^*/\phi$ with $z^* = \theta c(q^*) + (1-\theta)u(q^*)$. The condition $M < m^* = z^*/\phi$ should be interpreted here not as a restriction on M but as an equilibrium condition on the endogenous variable ϕ; that is, $\phi < z^*/M$, which gives an upper bound for ϕ. In any event, we can now define a *monetary equilibrium* for this model, with $m = M$ constant, in terms of the value function $V(m, s)$ satisfying the Bellman equation, the solution to the bargaining problem given by $\hat{q}(m)$ and $d(m) = m$, and a positive bounded path $\phi_{+1} = \Phi(\phi)$ such that the first order condition $\phi = \beta V'(m, s_{+1})$ holds at every date. Implicit in this definition is the distribution of $F(m)$, but we know it is degenerate.[13]

[12] Recall these conditions were either $U = C$ or the initial distribution F is not too disperse. Suppose these conditions are violated. Then some agent with a large endowment m will set $Y = 0$, $X > X^*$, and $m_{+1} \in (M, m)$. Hence, this agent will have above average money holdings for several periods, but eventually he will spend down until $Y(m) > 0$. After several periods, with probability 1, all agents will spend down until $Y(m) > 0$, after which F will be degenerate at $m = M$ for the rest of time.

[13] Of course, there is also always a nonmonetary equilibrium, where $\phi = q = 0$ for all t and $(1-\beta)V(m, s) = \alpha\delta b + U(X^*) - X^*$.

2.3 Results

In this subsection we show how to solve the model and characterize some aspects of the outcome. We begin by reducing the equilibrium conditions to one equation in one unknown. Inserting (13) into $\phi = \beta V'(m, \phi_{+1})$, we have

$$\phi = \beta[\alpha\sigma\phi_{+1}e(q_{+1}) + (1 - \alpha\sigma)\phi_{+1}]. \tag{17}$$

Rewrite this as $e(q_{+1}) = 1 + \frac{\phi - \beta\phi_{+1}}{\alpha\sigma\beta\phi_{+1}}$, then insert $\phi = \phi(q) = \frac{1}{M}\frac{\theta c u' + (1-\theta)uc'}{\theta u' + (1-\theta)c'}$ from (9), to get a difference equation in q:

$$e(q_{+1}) = 1 + \frac{\phi(q) - \beta\phi(q_{+1})}{\alpha\sigma\beta\phi(q_{+1})}. \tag{18}$$

A monetary equilibrium is now simply a solution to (18) such that $q \in [0, q^*]$ for all time.

While (18) may appear daunting, it simplifies dramatically in some cases. First, consider $\theta = 1$ (take-it-or-leave-it offers by buyers), a popular case in the literature. Since $\theta = 1$ implies $\phi(q) = q/M$ and $e(q) = u'(q)$, (18) reduces to

$$u'(q_{+1}) = 1 + \frac{q - \beta q_{+1}}{\alpha\sigma\beta q_{+1}}, \tag{19}$$

where we are using the normalization $c(q) = q$.[14] Second, regardless of θ, if we restrict attention to steady states (i.e., $q = q_{+1}$) then (18) reduces to

$$e(q) = 1 + \frac{1 - \beta}{\alpha\sigma\beta}. \tag{20}$$

[14] Without this normalization $\theta = 1$ implies $\phi(q) = c(q)/M$ and $e(q) = u'(q)/c'(q)$, and (19) would look only slightly more complicated.

For now we focus on steady states, and return to dynamics in the next section.

First consider steady states when $\theta = 1$, in which case (20) becomes

$$u'(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta}. \tag{21}$$

There obviously cannot be more than one solution to (21) since $u'' < 0$. Since our normalization implies $u'(q^*) = 1 < 1 + \frac{1-\beta}{\alpha\sigma\beta}$, a steady state, say q^s, exists iff $u'(0) > 1 + \frac{1-\beta}{\alpha\sigma\beta}$. Clearly, if a steady state exists then $q^s < q^*$. Also note that q^s is increasing in $\alpha\sigma$ and β, that $q^s \to q^*$ as $\beta \to 1$, and that $q^s \to 0$ as $\alpha\sigma \to 0$. Moreover, since $\theta = 1$ implies $\phi = q/M$, the nominal price of general and special goods in this case are both given by $p_g = p_s = M/q^s$. This implies *classical neutrality*: increasing M increases all nominal variables proportionately and leaves all real variables unchanged.

For general θ, note the following. First, it can be shown that $e(q)$ shifts down at any q solving (20) when θ decreases, and so $q < q^*$ for all θ. If $\theta = 0$ then there is no monetary steady state. For $\theta > 0$ a monetary steady state exists if $e(0) > 1 + \frac{1-\beta}{\alpha\sigma\beta}$. For example, if $u(q) = \frac{(b+q)^{1-\eta} - b^{1-\eta}}{1-\eta}$ where $\eta > 0$ and $b \in (0, 1)$, a monetary steady state exists iff

$$\theta > \frac{1-\beta}{\alpha\sigma\beta\,(b^{-\eta} - 1)}.$$

If q^s is unique then $\partial q/\partial\theta > 0$. For general θ we cannot be sure of uniqueness since we do not know the sign of e'; however, if u' is log concave one can show $e' < 0$ and hence uniqueness for any θ. For $\theta < 1$, q is bounded away from q^* even in the limit as $\beta \to 1$ (see below). Also, for any θ, q is still independent of M while $p_s = M/q$ and $p_g = M/\phi(q)$ are proportional to M.

Universidad de San Andrés

This completes the description of steady states. In the next section we study more complicated equilibria. Before doing so we will compare the model to others in the literature. We close this section by summarizing some things we have learned so far.

Proposition 1 *Any monetary equilibrium implies $m = M$ with probability 1 (F degenerate), implies the constraint $d \leq m$ holds with equality, and implies $q < q^*$ for all $t > 0$. Given any $\theta > 0$, a steady state $q^s > 0$ exists if $e(0) > 1 + \frac{1-\beta}{\alpha\sigma\beta}$. It is unique if $\theta \approx 1$ or u' is log concave, in which case q^s is increasing in β, $\alpha\sigma$ and θ. It converges to q^* as $\beta \to 1$ iff $\theta = 1$. Nominal variables are proportional to and real variables are independent of M.*

2.4 Related Literature

To compare our model to previous search models, we first shut down the production of general goods, so that there is no centralized market and all activity takes place in the decentralized market. The first generation of monetary search models made two severe assumptions: that individual money holdings are restricted to $m \in \{0, 1\}$, and that q is fixed exogenously. Letting V_m denote the value function and b_m the flow payoff (e.g., the expected gains from barter) of an agent holding $m \in \{0, 1\}$, the Bellman equations can be written

$$\begin{aligned} V_1 &= b_1 + \alpha\sigma(1-M)\left[u(q) + W_0\right] + \left[1 - \alpha\sigma(1-M)\right] W_1 \\ V_0 &= b_0 + \alpha\sigma M \left[W_1 - c(q)\right] + (1 - \alpha\sigma M) W_0, \end{aligned}$$

where here the value of leaving a match with m is simply $W_m = \beta V_m$ since there is no centralized market.

These equations essentially describe the model in Kiyotaki and Wright (1993). As long as two incentive conditions hold, $u(q) + W_0 \geq W_1$ and $W_1 - c(q) \geq W_0$, money circulates (is valued) in equilibrium. While there is not much that is endogenous in this model, it can at least be said that it formalizes why fiat money may be valued, how this can enhance efficiency, and so on.[15] Of course, an obvious drawback is that prices are exogenous, since every trade is a one-for-one swap. Second generation models kept $m \in \{0, 1\}$, but endogenized q by adding a bargaining solution,

$$\max_q [u(q) + W_0 - W_1]^\theta [-c(q) + W_1 - W_0]^{1-\theta}$$

(Shi [1995]; Trejos and Wright [1995]). In addition to demonstrating why money circulates, this model can be used to study prices, even though $m \in \{0, 1\}$.[16]

The third generation of search models allowed money holdings to be any $m \in \mathcal{M} \subset \mathbb{R}_+$. Given the distribution $F(\tilde{m})$, the Bellman equation is given by (1) and the bargaining solution is (5), both with $W(m) = \beta V(m)$, and we need to add a steady state condition for F. With $\mathcal{M} = \mathbb{R}_+$, this is

[15] For example, versions of this simple framework are applied to the study of commodity money in Kiyotaki and Wright (1989), to international currency in Matsuyama, Kiyotaki and Matsui (1993) and Zhou (1997), to the relation between money and private information in Williamson and Wright (1994), and to the optimal taxation of currency in Li (1995).

[16] For example, this model is applied to the relation between money and bonds in Aiyagari et al. (1996), to the Phillips' curve in Wallace (1997), to the relation between money and memory in Kocherlakota (1998), to Gresham's Law in Velde at al (1999), to inside versus outside money in Cavalcanti and Wallace (1999a,b), and to commodity money in Burdett et al. (2001).

Universidad de SanAndrés

the model in Molico (1999).[17] What makes our model so much simpler – i.e., what allows us to actually *solve* the model – is the presence of the centralized market in general goods. This does several things. First, it yields the linearity of W in m, which simplifies the Bellman equation and the bargaining solution considerably. Additionally, as all agents take the same m_{+1} out of the centralized market, in our model F is degenerate. That is, we have a representative agent in the decentralized market.

Closely related are the papers that follow Shi (1997), where there is also a degenerate distribution $F(\tilde{m})$, but for a very different reason.[18] These models assume the fundamental decision-making unit is not an individual but a family with a continuum of agents. Each household's agents search in a standard decentralized market, but at the end of each round they meet back at the homestead to share their money receipts net of expenditures. By a law of large numbers, each family has the same total money, and it divides it evenly among its members (or sometimes among a subset designated as buyers). Hence, in the next round every buyer has the same money holdings. The large-household "trick" is a similar device to our assumption of a centralized general goods market: both designed to make the model tractable by harnessing F.

While we think both approaches are useful, it seems incumbent upon us say why we like our "trick." First, some people seem to view the infinite

[17] For related models, see Green and Zhou (1997), Camera and Corbae (1999), Taber and Wallace (1999), Zhou (1999), and Berentsen (2001).

[18] See also Shi (1998,1999), Head and Shi (2000), Rauch (2000), Berentsen and Rocheteau (2000a,b), Berentsen, Rocheteau and Shi (2001), Faig (2001), and Head and Kumar (2001).

family structure as unappealing (perhaps unrealistic?) and on these grounds tend to dismiss it. While we do not necessarily agree with this view – after all, Shi's families are just logical extensions of Lucas's (1980) worker-shopper households – it seems important to have an alternative lest anyone think that tractable monetary models with search-theoretic foundations require infinite families. Second, there are complications that arise in family models due to the fact that infinitesimal agents bargain over trades that benefit not themselves but the larger family unit (see Rauch [2000]). This is not the case in our framework, where individuals bargain for themselves, and so we can use standard bargaining theory. Indeed, the linearity of $W(m)$ makes our bargaining solution extremely simple.

Third is the related but distinct point that individual incentive conditions are typically not taken into account in family models: individual agents act not in their own self interest, but according to rules prescribed by the head of the household. Thus, every time an agent produces in order to acquire cash he suffers a cost, but in principle he could report back to the clan without cash and claim he had no customers. This would save the cost with no implication for his future payoff. For the family structure to survive, agents must act in the interest of the household and not themselves. In our model, by contrast, individual incentive constraints are always taken into account. Finally, we simply find the general goods model more transparent and easier to use. However, we repeat that we think both approaches are useful, and a choice may come down to taste and to the application at hand.

3 Extensions

3.1 Policy

Suppose the money supply grows at a constant rate, $M_{+1} = (1+\tau)M$. The new money is injected as a lump-sum transfer, or tax if $\tau < 0$, that occurs after agents leave the centralized market. The Bellman's equation becomes

$$V(m, \phi) = \max_{m_{+1}} \left\{ v(m, \phi) + \phi m - \phi m_{+1} + \beta V \left(m_{+1} + \tau M, \phi_{+1} \right) \right\}$$

where v is defined in (12) and we write $s = \phi$ since F will be degenerate. The agent takes m_{+1} out of the centralized market knowing he will receive his transfer τM before the decentralized market opens next period. In this subsection we maintain attention on steady states, where q is constant. This means real balances $z = \phi M$ are also constant, and therefore $\phi_{+1} = \phi/(1+\tau)$. Exactly as in the previous section, $\phi_t \geq \beta \phi_{+1}$ is necessary for an equilibrium to exist (Lemma 3); hence we require the constraint $\tau \geq \beta - 1$ on monetary policy.

The first order condition for m_{+1} is $\phi = \beta V_1 \left(m_{+1} + \tau M, \phi_{+1} \right)$. Inserting V_1 and rearranging, we arrive at the generalized version of (20),

$$e(q) = 1 + \frac{1 - \beta + \tau}{\alpha \sigma \beta}. \tag{22}$$

Assuming a unique steady state exists, $\partial q^s / \partial \tau = 1/\alpha \sigma \beta e' < 0$ (of course, if there are multiple steady states the derivative alternates in sign). From (22) it appears that all one needs to do to achieve the efficient outcome $q^s = q^*$ is to set

$$\tau^* = \beta - 1 - \alpha \sigma \beta \left[1 - e(q^*) \right].$$

However, we have already established that there is no equilibrium if $\tau < \beta - 1$, and in general we need to take this feasibility constraint into account.

If $\theta = 1$ then $e(q^*) = 1$, and it is feasible to set $\tau = \tau^* = \beta - 1$; i.e., $q^s = q^*$ obtains if we adopt what is called the *Friedman rule* and deflate at the rate of time preference. If $\theta < 1$, however, then $e(q^*) < 1$, and at the lowest feasible $\tau = \beta - 1$ we still have $q^s < q^*$. Hence, the Friedman rule always maximizes q^s, which is optimal, but achieves full efficiency iff $\theta = 1$. The reason is that the model has two types of inefficiencies, one due to β and one to θ. To describe the first effect, note that when you accept cash you get a claim to future consumption, and because $\beta < 1$ you are willing to produce less for cash than the q^* you would produce if you could turn it into immediate consumption (recall $q^s \rightarrow q^*$ as $\beta \rightarrow 1$ when $\theta = 1$). The Friedman rule simply generates a rate of return on money due to deflation that compensates for discounting.

The wedge due to $\beta < 1$ is standard, and the only difference from, say, a cash-in-advance model on this dimension is that here the frictions show up explicitly: (22) shows that for a given β and τ the inefficiency gets worse as $\alpha\sigma$ gets smaller. The other, more novel, effect is the wedge due to $\theta < 1$. One intuition for this effect is the notion of a *hold-up problem.* Think of an agent who carries a dollar into next period as making an investment, with cost ϕ. When he spends the money he reaps all of the returns to his investment iff $\theta = 1$; otherwise the seller "steals" part of the surplus. Thus $\theta < 1$ reduces the incentive to invest, which lowers the demand for money and hence q, and therefore $\theta < 1$ implies $q^s < q^*$ even at the Friedman rule.[19] We conclude that monetary policy can undo the β-wedge but not the θ-wedge.

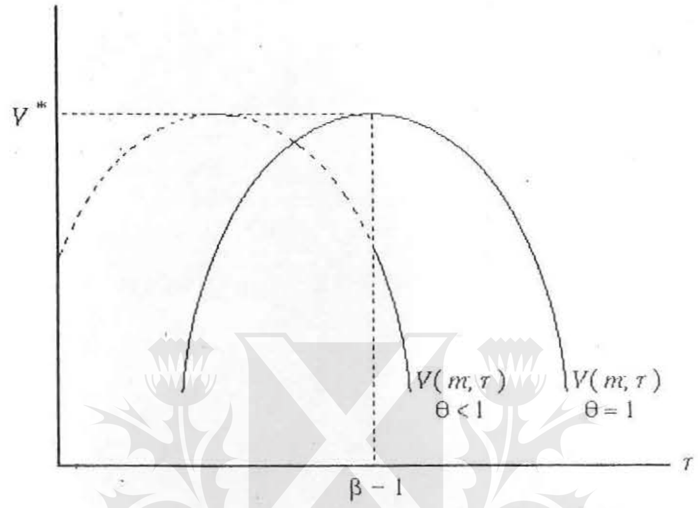

Figure 4: Welfare Cost of Inflation

This is something that obviously does not come up in the usual reduced-form model, and it may be important. Consider the welfare cost of inflation. When $\theta = 1$, welfare, as measured by the payoff of the representative agent V, is maximized at $\tau = \beta - 1$ and achieves the efficient solution

$$V^* = \frac{\alpha(\delta + \sigma)\left[u(q^*) - c(q^*)\right] + U(X^*) - X^*}{1 - \beta}.$$

See Figure 4. Small deviations from $\tau = \beta - 1$ clearly have very small effects on welfare by the Envelope Theorem. When $\theta < 1$, $\tau = \tau^* < \beta - 1$ would

[19] Recall Hosios' (1990) general condition for efficiency in search models, which bascially says the bargaining outcome should split the gains from trade so as to compensate each party for his contribution to the match-specific surplus. Here the match-specific surplus is totally due to the buyer (recall the bargaining solution depends on his money holdings and not those of the seller). Hence, efficiency requires $\theta = 1$.

achieve V^*, but it is not feasible. At the constrained optimum the slope of V with respect to τ is steep, so a moderate inflation will have a much bigger effect. Of course, it will be important ultimately to calibrate the model carefully to see just how big this effect is.

3.2 Dynamics

In many monetary economies, even if the environment is stationary, there can be equilibria other than steady states; so here we consider dynamics. We also introduce a real flow return γ per nominal unit of money, since this has an interesting impact on the set of equilibria and allows one to make some additional points. One can interpret this as a dividend or interest on currency if $\gamma > 0$; as a storage cost or tax if $\gamma < 0$; and as the case of pure fiat money if $\gamma = 0$. Note than γ is a real return per unit of nominal currency m (a return on real balances z gives less interesting results). Here we assume M and all other exogenous variables are constant, and consider for now equilibria where q may change over time due to self-fulfilling expectations. We also normalize $c(q) = q$ and set $M = 1$ without loss in generality, and assume $\theta = 1$ unless otherwise indicated.

These assumptions imply $\phi = q$, so we do all the dynamics in terms of q in this section. Then, taking into account the flow return γ, condition (17) becomes

$$q = \beta[\alpha\sigma q_{+1} u'(q_{+1}) + (1 - \alpha\sigma) q_{+1} + \gamma]. \tag{23}$$

Rather than rearranging this as before to get a version of (19), here we simply

BIBLIOTECA UNIVERSIDAD DE SAN ANDRES MAX VON BUCH

denote the right hand of by $G(q_{+1})$. The dynamic model is closed by setting $G^{-1} = \Phi$, where $\phi_{+1} = \Phi(\phi)$ is the law of motion agents took as given above, since in this case $q = \phi$.[20] An equilibrium is now a solution to difference equation (23) that stays in the bounded set $[0, q^*]$. Figure 5 shows examples of the function $q_{+1} = \Phi(q)$ for various values of γ.

Consider the case where $\gamma = 0$. Assuming $u'(0) > 1 + \frac{\beta-1}{\alpha\sigma\beta}$, we know from the previous section that there exists a unique steady state $q^s > 0$. One can show that $G(0) = 0$ and $G'(0) > 1$, and so, as seen in the Figure, q^s is unstable.[21] This means that there exists a continuum of dynamic equilibria: starting at any $q_0 \in (0, \bar{q}_0)$, where $\bar{q}_0 \geq q^s$, there is a path starting at q_0 and staying in $[0, q^*]$ such that $q \to 0$. Therefore, exactly as in many other models of fiat money, even if all fundamentals including the money supply are time invariant, expectations of inflation can be self-fulfilling. There may also be dynamic equilibria that do not converge to $q = 0$, as we discuss below, but first we want to consider the effect of allowing $\gamma \neq 0$.

Consider $\gamma < 0$. As compared to the case $\gamma = 0$, this means shifts $\Phi = G^{-1}$ up uniformly in Figure 5. It is easy to see that there is some $\underline{\gamma} < 0$

[20] A complcation arises due to the fact that G may not be invertible (i.e., G^{-1} may not be single-valued). In the Appendix we discuss how one can restrict attention to a certain class of equilibria where we select $\phi_{+1} = \Phi(\phi)$ from the correspondence G^{-1} in a continuous way. This allows the use of dynamic programming (we need Φ to define the law of motion for the aggregate state in our Bellman equation), but does not otherwise matter for what we do; it should be clear that the dynamics can all be understood from G whether or not $\Phi = G^{-1}$ is a function.

[21] By concavity, $0 \leq u'(q)q \leq u(q)$ for all q. Hence $u'(q)q \to 0$ as $q \to 0$ and so $G(0) = \beta\gamma$. It is easy to show that $G'(q) = [G(q) - \beta\gamma]/q + \beta\alpha\sigma q u''(q)$ and hence $G'(q^s) = 1 - \beta\gamma/q^s + \beta\alpha\sigma q^s u''(q^s) < 1$ at any steady state $q^s > 0$ when $\gamma \geq 0$. This implies $G(q)$ cuts the 45^o line from above at q^s, and therefore there can be at most one positive steady state when $\gamma \geq 0$. This also implies $G'(0) > 1$.

UNIVERSIDAD DE SAN ANDRES
MAX VON BUCH

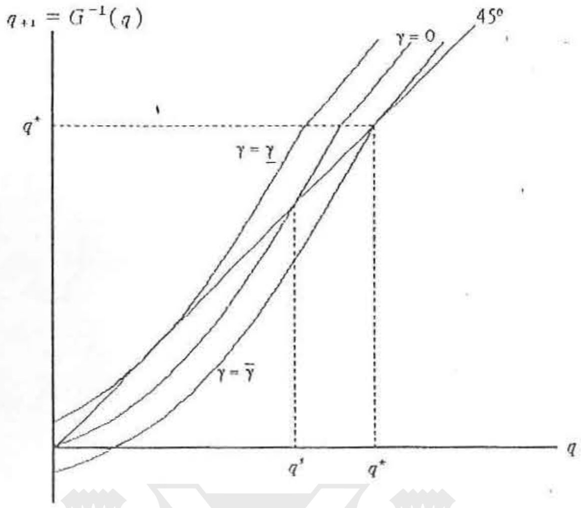

Figure 5: $\phi = G^{-1}(\phi_{+1})$ for different γ

Universidad de San Andrés

such that if $\gamma < \underline{\gamma}$ then there exist no monetary equilibria: starting from any $q_0 > 0$, the path $q_{+1} = \Phi(q)$ diverges. Hence, if money has very bad intrinsic properties, like a very high storage cost, there can be no equilibrium where it is valued – steady state or otherwise. If $\gamma \in (\underline{\gamma}, 0)$ then there are generically an even number of monetary steady states ($u''' < 0$ implies G is concave and there are exactly 2 monetary steady states). Clearly all steady states have q below q^s, the steady that prevailed when $\gamma = 0$. As should also be clear, alternate steady states are stable and unstable. Hence, if the intrinsic properties of money are bad but not too bad (γ negative but less $\underline{\gamma}$ in absolute value) there are multiple monetary equilibria that can converge to either a monetary steady state or to $q = 0$, depending on q_0.

Now consider $\gamma > 0$. As compared to $\gamma = 0$, this means $\Phi = G^{-1}$ shifts down in Figure 5. There still exists a unique steady state, but now there are no other bounded paths satisfying $q_{+1} = \Phi(q)$; hence there is no equilibrium other than the monetary steady state.[22] We cannot be sure that $q < q^*$ in equilibrium when $\gamma > 0$; in fact, one can show $q < q^*$ iff $\gamma < \overline{\gamma} = \frac{1-\beta}{\beta} q^*$ (given $M = 1$). When $\gamma \geq \overline{\gamma}$, the bargaining solution implies $q = q^*$ and $d < m$ – that is, the real value of money is sufficiently high that the constraint $d \leq m$ does not bind, and hence the unique equilibrium is efficient. It may therefore seem like a good idea for money to pay interest, but note that satisfying $\gamma \geq \overline{\gamma}$ can be expensive if we have to somehow finance the interest payments. In any case, the model with $\gamma > 0$ has no equilibria other than the steady state.

We now return to $\gamma = 0$, and show how more complicated dynamic outcomes are possible. When $\gamma = 0$ there is a unique monetary steady state $q^s < q^*$, and $G'(q^s) = 1 + \beta\alpha\sigma q u''(q^s)$. Figure 5 depicted a case where $G'(q^s) > 0$, but we can also have $G'(q^s) < 0$. Indeed, we have $G'(q) < -1$ iff $qu''(q) < -2/\beta\alpha\sigma$. Figure 6 shows $q = G(q_{+1})$ and $q_{+1} = \Phi(q)$, which intersect on the 45^o line at q^s, in a case where $G'(q) < -1$. Clearly, this implies G and Φ must also intersect off the 45^o line: since Φ is less steep

[22] Additionally, suppose we relax the assumption $u'(0) > 1 + \frac{\beta-1}{\alpha\sigma\beta}$, so that monetary equilibria do not exist when $\gamma = 0$. By giving money any strictly positive dividend γ, no matter how small, we eliminate the nonmonetary equilibrium and guarantee the existence of a unique monetary equilibrium (and this is true for all $\theta > 0$, not just $\theta = 1$). One way to see how things work is to note that the generalized version of (20) is

$$e(q) = 1 + \frac{1-\beta}{\alpha\sigma\beta} - \frac{\gamma}{\alpha\sigma q},$$

which always has a solution $q^s > 0$. For similar results in a model where $m \in \{0, 1\}$, see Li and Wright (1998).

than G at q^s, G is trapped between the two branches of Φ to the right of q^s. The figure shows Φ crossing G at $(q, q_{+1}) = (q_H, q_L)$. This generates an equilibrium 2-cycle: $q = q_L$ implies $q_{+1} = \Phi(q_L) = q_H$ and $q = q_H$ implies $q_{+1} = \Phi(q_H) = q_L$. Equivalently, q_L and q_H are fixed points of the second iterate G^2 (see Azariadis [1993] for an exposition of the mathematics behind these results).

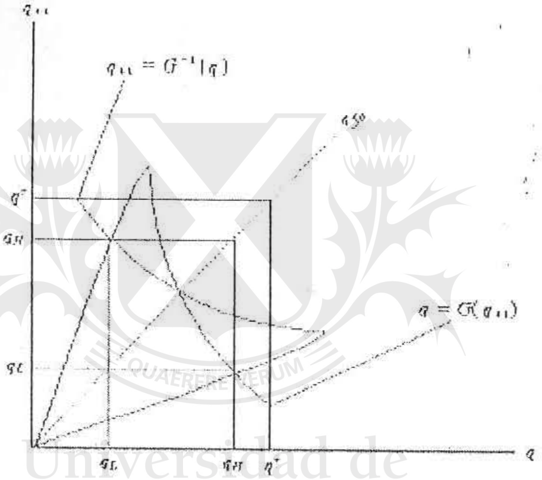

Figure 6: Dynamics: Two Cycle

Universidad de San Andrés

To give an example, consider

$$u(q) = \frac{(b+q)^{1-\eta} - b^{1-\eta}}{1-\eta},$$

where $b \in (0,1)$ and $\eta > 0$. Then

$$q^s = \left(1 + \frac{\beta - 1}{\beta\alpha\sigma}\right)^{-\frac{1}{\eta}} - b.$$

For $b \approx 0$ we have $G'(q) \approx 1 - \eta(1 - \beta + \beta\alpha\sigma)$, and so $G'(q) < -1$ iff $\eta > \frac{2}{1-\beta+\beta\alpha\sigma}$. More generally, given $b > 0$ there is a critical $\bar{\eta}$ such that as we increase η beyond $\bar{\eta}$ the system bifurcates and a 2-cycle emerges. As η increases further, cycles of other periodicity emerge. With $\alpha = 1$, $\sigma = 0.5$, $b = 0.01$, and $\beta = 1/1.1$, we found cycles of period 3 emerge at $\eta \approx 3.9$ (this is verified by looking for fixed points of G^3 different from q^s). Once we have cycles of period 3, we have cycles of all periods (again, see Azariadis [1993]). Hence, like some other models of money, this model is capable of generating some quite interesting dynamics.

3.3 Real Uncertainty

So far we have dealt with a deterministic environment, and within that setting the constraint $d \leq m$ is binding at every date. We now extend the model to allow for stochastic tastes and technology and show that the constraint may not always bind. We begin with a model with match-specific uncertainty: when any two agents meet they draw a random $\varepsilon = (\varepsilon_b, \varepsilon_s) \geq 0$ from $H(\varepsilon)$, independently across meetings, and the implied utility of consumption and cost of production in that meeting are then given by $\varepsilon_b u(q)$ and $\varepsilon_s q$. For simplicity, we set $c(q) = q$ and $\theta = 1$.

Letting $q = q(m, \varepsilon)$ and $d = d(m, \varepsilon)$ denote the bargaining outcome, (7) becomes

$$q(m,\varepsilon) = \begin{cases} \widehat{q}(m,\varepsilon) & \text{if } m < m^*(\varepsilon) \\ q^*(\varepsilon) & \text{if } m \geq m^*(\varepsilon) \end{cases} \quad \text{and } d(m,\varepsilon) = \begin{cases} m & \text{if } m < m^*(\varepsilon) \\ m^*(\varepsilon) & \text{if } m \geq m^*(\varepsilon) \end{cases}$$

where $\widehat{q}(m,\varepsilon) = \phi m/\varepsilon_s$, $q^*(\varepsilon)$ solves $u'(q) = \varepsilon_s/\varepsilon_b$, and $m^*(\varepsilon) = \varepsilon_s q^*(\varepsilon)/\phi$.

In the deterministic case, when agents choose how much money to take to the decentralized market they know whether this will be enough to afford q^*. Now, for a given realization of ε_s, buyers with high realizations of ε_b will spend all their cash but those with low ε_b will not. Let $C \doteq \{\varepsilon | q^*(\varepsilon) > \phi m/\varepsilon_s\}$ be the set of realizations such that $d \leq m$ is binding. Also, let $b(\varepsilon)$ denote the gain from trade in a double coincidence meeting given ε and $Eb = \int b(\varepsilon) dH(\varepsilon)$.[23]

We can still write the Bellman equation exactly as in (11), but now

$$v(m,s) = \alpha\sigma \int \{\varepsilon_b u[q(m,\varepsilon)] - \phi d(m,\varepsilon)\} \, dH(\varepsilon) + \alpha\delta Eb + U(X^*) - X^* \tag{24}$$

Notice that $v_1 = \alpha\sigma\phi \int_C [\varepsilon_b u'(\phi m/\varepsilon_s)/\varepsilon_s - 1] \, dH(\varepsilon) > 0$, since $q = \phi m/\varepsilon_s$ and $d = m$ on C while q and d are constant on C^c. Clearly, $v_{11} < 0$, and so the distribution of money F is still degenerate at $m_{+1} = M$, exactly as in the deterministic case. The first order condition for m_{+1} is $\beta V_1(m_{+1}, s_{+1}) = \phi$, which after inserting $V_1 = v_1 + \phi$ implies a generalized version of (21):

$$\int_C \left[\left(\frac{\varepsilon_b}{\varepsilon_s}\right) u'\left(\frac{M\phi_{+1}}{\varepsilon_s}\right) - 1 \right] dH(\varepsilon) = \frac{\phi - \beta\phi_{+1}}{\alpha\sigma\beta\phi_{+1}}. \tag{25}$$

It is easier here to consider solutions to (25) in terms of ϕ rather than in terms of q in the current application, simply because q is now a function that depends on the realization of ε in each meeting while ϕ is a number. It is obvious that if u satisfies the usual Inada conditions then there exists a unique monetary steady state, $\phi^s > 0$. Given ϕ^s, the match-specific values of

[23] We assume ε applies to both agents in the meeting, and so the symmetric Nash solution implies both agents produce $q^*(\varepsilon)$ and no money changes hands.

$q = q(M, \varepsilon)$ and $d = d(M, \varepsilon)$ are obtained from the bargaining solution. The match-specific nominal price of a special good is given by $p_s(\varepsilon) = \varepsilon_s / \phi\phi^s$, independent of whether the constraint binds (it is also independent of ε_b, simply because $\theta = 1$ in this example). Classical neutrality still holds. Finally, we remark that in any equilibrium the constraint $d \leq m$ must bind with positive probability, since otherwise C is empty and (25) could not hold, but it is possible for it to bind with probability 1 or to bind with probability less than 1.[24]

So far we have been interpreting the shock as *idiosyncratic* to a match, but the same analysis applies if ε is an *aggregate* shock to tastes and technology, as long as it is i.i.d. – exactly the same equations and results apply. We now consider the case where ε is not i.i.d., in which case we need to interpret it as an aggregate shock. Let $H(\varepsilon_{+1}|\varepsilon)$ denote the conditional distribution of ε_{+1} given ε. The bargaining solution is the same as in the i.i.d. case, but now

QUAERERE VERUM
Universidad de
San Andrés

$$V(m, \varepsilon) = \max_{m_{+1}} \left\{ v(m, \varepsilon) + \phi m - \phi m_{+1} + \beta \int V(m_{+1}, \varepsilon_{+1})\, dH(\varepsilon_{+1}|\varepsilon) \right\}$$

where

$$v(m, \varepsilon) = \alpha\sigma \left\{ \varepsilon_b u\left[q(m, \varepsilon)\right] - \phi d(m, \varepsilon) \right\} + \alpha\delta b(\varepsilon) + U(X^*) - X^*$$

[24] The constraint binds iff $q^* > \phi m / \varepsilon_s$, which in turn holds iff $\varepsilon_b > \overline{\varepsilon}_b \equiv \varepsilon_s / u'(\phi m / \varepsilon_s)$. Notice

$$\frac{\partial \overline{\varepsilon}_b}{\partial \varepsilon_s} = \frac{1 - R(\phi m / \varepsilon_s)}{u'(\phi m / \varepsilon_s)},$$

where $R(q) = -u''(q)\, q / u'(q)$; hence we cannot be sure of the sign. In any case, suppose $\varepsilon_s = 1$ with probability 1 and ε_b is uniform on $[0,1]$. Then $\overline{\varepsilon}_b$ satisfies $\beta \left[1 + \frac{\alpha\sigma}{2\overline{\varepsilon}_b}(1 - \overline{\varepsilon}_b)^2\right] = 1$. There is a unique $\overline{\varepsilon}_b$ that solves the equation, and it lies in $(0, 1)$. Hence, the constraint binds with probability $1 - \overline{\varepsilon}_b \in (0, 1)$.

and $b(\varepsilon)$ now applies to all double-coincidence meetings when the aggregate state is ε. Note that ε can in principle affect the gains from trade in both barter and monetary exchange, as well as the forecast of $V(m_{+1}, \varepsilon_{+1})$.

Generalizing the argument from the i.i.d. case, we have $v_1(m, \varepsilon) = \alpha\sigma[\phi\varepsilon_b u'(\phi m/\varepsilon_s)/\varepsilon_s - 1]$ if $\varepsilon \in C$, and $v_1 = 0$ if $\varepsilon \in C^c$. Also, $v_{11} < 0$ for $\varepsilon \in C$, so $\beta \int V(m_{+1}, \varepsilon_{+1}) dH(\varepsilon_{+1}|\varepsilon) - \phi m_{+1}$ is strictly concave, and again all agents again choose the same m_{+1}, given by $\phi = \beta \int V_1(m_{+1}, \varepsilon_{+1}) dH(\varepsilon_{+1}|\varepsilon)$. Suppose we look for equilibria where $\phi = \phi(\varepsilon)$ is a stationary function of the aggregate shock (the analogue of a steady state). Then, after inserting V_1 into the first order condition, we find

$$\phi(\varepsilon) = \beta \int \phi(\varepsilon_{+1}) \left(1 + I(\varepsilon_{+1}) \alpha\sigma \left\{ \frac{\varepsilon_{+1,b}}{\varepsilon_{+1,s}} u' \left[\frac{\phi(\varepsilon_{+1}) M}{\varepsilon_{+1,s}} \right] - 1 \right\} \right) dH(\varepsilon_{+1}|\varepsilon) \tag{26}$$

where $I(\varepsilon)$ is an indicator function that equals 1 if $\varepsilon \in C$ and 0 otherwise.

In general, (26) is a functional equation in $\phi(\cdot)$. We will show by way of example how to solve such an equation in the next section (with monetary rather than real shocks, but the method is the same). Here we simply note that for the special case where the shocks are i.i.d. the right hand side of (26) is independent of ε, and hence $\phi(\varepsilon)$ is constant. The key economic point is that the current state ε affects the value of money today only to the extent that it enters the forecast of next period's state. Still, to the extent that real shocks are not i.i.d., there will be feedback from ε to the value of money, and it may be interesting in future work to analyze the empirical implications further.

3.4 Monetary Uncertainty

Let us return now to the case where $\varepsilon_b = \varepsilon_s = 1$ with probability 1, and introduce uncertainty in the money supply. We first consider random transfers of money across agents, so that we can easily talk about risk and the "precautionary" demand for money, and then take up transfers that are uniform across agents but random over time. For the first experiment, suppose that each period before the start of trade money is randomly redistributed so that an agent who brought m dollars into the period enters the decentralized market with $m + \rho$ dollars, where ρ has CDF $H(\rho)$. For now we assume $E\rho = 0$, so that the total money stock M is constant, and the support of ρ is $\left[\underline{\rho}, \overline{\rho}\right]$, with $\underline{\rho} \geq -M$. Here we also set $\theta = 1$.

The value function again satisfies (11), where now

$$v(m, s) = \alpha\delta b + U(X^*) - X^* + \alpha\sigma \int \{u[q(m+\rho)] - \phi d(m+\rho)\}\, dH(\rho).$$

The bargaining solution is given by $q(m+\rho)$ and $d(m+\rho)$, where the functions $q(\cdot)$ and $d(\cdot)$ are exactly as in (7). Notice that

$$v_1(m, s) = \alpha\sigma\phi \int_{\underline{\rho}}^{\hat{\rho}} \{u'[(m+\rho)\phi] - 1\}\, dH(\rho),$$

where $\hat{\rho} = q^*/\phi - m$ is the minimum transfer that makes the constraint slack. As above, $v_{11} < 0$, so all agents choose the same m_{+1}, and the usual procedure yields

$$\int_{\underline{\rho}}^{q^*/\phi_{+1} - M} \{u'[(M+\rho)\phi_{+1}] - 1\}\, dH(\rho) = \frac{\phi - \beta\phi_{+1}}{\alpha\sigma\beta\phi_{+1}}. \tag{27}$$

Universidad de San Andrés

Inada conditions imply exists a unique monetary steady state ϕ^s.

To analyze the effect of risk, consider a family of distributions $H(\rho,\Sigma)$ where $\Sigma_2 > \Sigma_1$ implies $H(\rho,\Sigma_2)$ is a mean preserving spread of $H(\rho,\Sigma_1)$: that is, $\Xi(\tilde{\rho},\Sigma) = \int_{\underline{\rho}}^{\tilde{\rho}} H_2(\rho,\Sigma)\,d\rho \geq 0$ for any $\widetilde{\rho}$ with equality at $\widetilde{\rho} = \bar{\rho}$. Notice that $H_2(\underline{\rho},\Sigma) = H_2(\overline{\rho},\Sigma) = 0$. Then $\partial\phi^s/\partial\Sigma$ is equal in sign to

$$\begin{aligned}\Psi &= \int_{\underline{\rho}}^{\hat{\rho}(\phi)} \{u'[(M+\rho)\phi] - 1\}\, dH_2(\rho,\Sigma) \\ &= [\{u'[(M+\rho)\phi] - 1\} H_2(\rho,\Sigma)]_{\underline{\rho}}^{\hat{\rho}} - \int_{\underline{\rho}}^{\hat{\rho}(\phi)} \phi u''[(M+\rho)\phi] H_2(\rho,\Sigma) d\rho,\end{aligned}$$

where the last expression results from integrating by parts. The first term vanishes because $u' = 1$ at $\hat{\rho}$ and $H_2(\underline{\rho},\Sigma) = 0$. Integrating by parts again,

$$\Psi = -\phi u''(q^*)\,\Xi(\hat{\rho},\Sigma) + \int_{\underline{\rho}}^{\hat{\rho}(\phi)} \phi^2 u'''[(M+\rho)\phi]\,\Xi(\rho,\Sigma)\,d\rho.$$

The first term is unambiguously positive but the sign of the second depends on u'''; if $u''' \geq 0$ then $\Psi > 0$ and a mean-preserving spread of H unambiguously increases the value of money.

The effect of Σ on ϕ^s and the way it depends on u''' is due to a "precautionary" demand for money. Simply put, given $u''' \geq 0$ an increase in risk makes agents want to hold more cash, which raises its value. It can also be shown that an increase in risk unambiguously reduces welfare.[25] This is

[25] To see this, write

$$\begin{aligned}V(m) &= \alpha\delta b + W(m) + \\ &\alpha\sigma \int (I(\rho)\{u[\phi(m+\rho)] - \phi(m+\rho)\} + [1 - I(\rho)][u(q^*) - q^*])\, dH(\rho,\Sigma),\end{aligned}$$

to be contrasted with some other models, where the distribution of money holdings F is non-degenerate in equilibrium and random transfers of across agents may be welfare improving (Molico [1999]; Berentsen [1999]). The reason is that random transfers can make the distribution of real balances less unequal, and in this way they provide partial insurance. Here, the distribution of real balances is degenerate, so random transfers cannot help on this dimension.

We now let the growth rate of the money supply τ be random with distribution $H(\tau_{+1}|\tau)$, with transfers arriving between centralized and decentralized trade, as above. Then

$$V(m,\tau) = \max_{m_{+1}}\{v(m,\tau) + \phi m - \phi m_{+1} + \beta \int V(m_{+1} + \tau_{+1}M, \tau_{+1})\, dH(\tau_{+1}|\tau)\}$$

where

$$v(m,\tau) = \alpha\delta b + U(X^*) - X^* + \alpha\sigma\{u[q(m)] - \phi d(m)\}.$$

Again let $C = \{\tau|(m+\tau M)\phi < q^*\}$ be the set of realizations where the constraint binds. Again all agents again choose the same m_{+1}, which now implies

$$\phi = \beta \int_{C^c} \phi_{+1} dH(\tau_{+1}|\tau) + \beta \int_{C} \phi_{+1}\left[\alpha\sigma u'(\phi_{+1}m_{+1}) + 1 - \alpha\sigma\right] dH(\tau_{+1}|\tau).$$

where $I(\rho)$ is an indicator function which equals 1 if $\rho < \hat{\rho} = q^*/\phi - M$ and 0 otherwise. Since the integrand is strictly concave in ρ, a mean-preserving spread unambiguously reduces $V(m)$.

If we focus on stationary equilibrium, then we can write

$$z(\tau) = \beta \int_{C^c} \frac{z(\tau_{+1}) dH(\tau_{+1}|\tau)}{1+\tau_{+1}} + \beta \int_{C} \frac{\{\alpha\sigma u'[z(\tau_{+1})] + 1 - \alpha\sigma\} z(\tau_{+1}) dH(\tau_{+1}|\tau)}{1+\tau_{+1}} \tag{28}$$

where $z = \phi m$. This is a functional equation in $z(\cdot)$. If shocks are i.i.d. then $z(\tau)$ is constant, and the constraint binds in every period. In this case

$$u'(z) = 1 + \frac{\zeta^{-1} - \beta}{\beta\alpha\sigma}$$

where $\zeta = \int (1+\tau)^{-1} dH(\tau)$ is the expected (gross) return to holding a unit of money for a period. A necessary condition for existence is $\zeta \leq 1/\beta$, which generalizes what we saw with deterministic money growth, $1 + \tau \geq \beta$.

When τ is persistent, the inflation forecast will depend on the current state, and hence so will real balances. Consider the following example. Suppose $\tau \in \{\tau_1, \tau_2\}$, with $\tau_1 > \tau_2$, $prob(\tau = \tau_i|\tau_i) = p_i$, $prob(\tau = \tau_1|\tau_2) = s_2$ and $prob(\tau = \tau_2|\tau_1) = s_1$. The process is i.i.d. if $p_1 = s_2$, and persistent if $p_1 > s_2$. In this case we can write (28) as two equations in (z_1, z_2), and we look for a solution (z_1^*, z_2^*) such that $z_i^* < q^*$. These two equations can be rearranged to solve for z_1 as a function of z_2 and vice-versa:

$$\begin{aligned} z_1 &= z_1(z_2) = \left[\frac{p_1}{s_2} - \frac{\beta(1-\alpha\sigma)(p_1 p_2 - s_1 s_2)}{s_2(1+\tau_2)}\right] z_2 - \frac{\beta\alpha\sigma(p_1 p_2 - s_1 s_2)}{s_2(1+\tau_2)} u'(z_2) z_2 \\ z_2 &= z_2(z_1) = \left[\frac{p_2}{s_1} - \frac{\beta(1-\alpha\sigma)(p_1 p_2 - s_1 s_2)}{s_1(1+\tau_1)}\right] z_1 - \frac{\beta\alpha\sigma(p_1 p_2 - s_1 s_2)}{s_1(1+\tau_1)} u'(z_1) z_1. \end{aligned}$$

These functions are shown in Figure 7.

Notice $z_i(0) = 0$ and $\lim_{z \to \infty} z_i(z) = \infty$. It may be shown that $z_i'(z) > 0$ as long as $R(z) = -u''(z) z / u'(z) \geq 1$. Let $\bar{z}_i = z_i(\bar{z}_i)$ be the point where the

Universidad de San Andrés

$z_i(\cdot)$ function crosses the 45^o line (see the Figure), given by the solutions to

$$\begin{aligned} 1+\alpha\sigma\left[u'\left(\overline{z}_2\right)-1\right] &= \frac{\left(1+\tau_1\right)\left(p_2-s_1\right)}{\beta\left(p_1p_2-s_1s_2\right)} \\ 1+\alpha\sigma\left[u'\left(\overline{z}_1\right)-1\right] &= \frac{\left(1+\tau_2\right)\left(p_2-s_1\right)}{\beta\left(p_1p_2-s_1s_2\right)}. \end{aligned}$$

These imply $\overline{z}_2 < \overline{z}_1$ if $\tau_1 > \tau_2$, and given $z_i'(\cdot) > 0$ this implies $z_1^* < z_2^*$. Hence, when the shocks to the money supply are persistent real balances are smaller in periods of high inflation. To conclude the example, recall that the equilibrium was constructed conjecturing that $z_i^* < q^*$. Since $z_i^* < \overline{z}_1$; for the conjecture to be correct it is sufficient to ensure that $\overline{z}_1 \leq q^*$, which holds iff $\left(1+\tau_2\right)\left(p_1-s_2\right) \geq \beta\left(p_1p_2-s_1s_2\right)$.

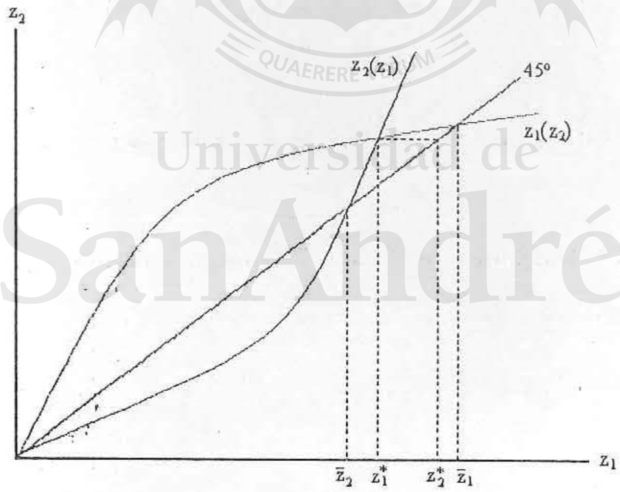

Figure 7: Equilibrium with Random τ

4 Conclusion

We presented a model based explicitly on the frictions that make money essential, as in the search-based monetary literature, but we relaxed an extreme assumption usually made in that literature – that agents can store very restricted inventories of money, like $m \in \{0, 1\}$. That restriction makes those models ill-suited for some issues, especially those involving changes in the money supply, but most versions without it are extremely complicated. The key innovation here is a centralized market that convenes between rounds of decentralized trade. Under certain assumptions on preferences all agents take the same amount of money out of the centralized market. The resulting framework has some desirable features of search models, including decentralized trade and price setting in specialized markets, and at the same time allows for fairly general monetary policies, as do reduced-form models. In this sense it helps to integrate micro and macro strands in monetary economics.

We were able to establish a fairly complete characterization of equilibria. In particular, at least in the basic version of the model with a constant money supply and no real shocks, the constraint $d \leq m$ holds with equality, and $q < q^*$ at all dates, in any equilibrium. Thus, monetary equilibria are inefficient. We provided existence and uniqueness results for stationary equilibria, and demonstrated how some interesting nonstationary equilibria are also possible. We also extended the analysis to allow for real or monetary shocks. We showed that all nominal (real) variables are proportional to (independent) of the level of M. In terms of the growth rate in M, we showed that the best policy is Friedman rule (deflate at the rate of time preference), but that this policy achieves the efficient allocation q^* iff the buyer has all the bargaining power. We argued that this could have important implications for thinking about the welfare cost of inflation.

Much more could be done. A quantitative analysis of the welfare cost of inflation, or of the effects of real and monetary shocks is beyond the scope of this project, but seems like a natural next step. One may want to perform such an analysis on an extended version of the model that endogenizes search intensity (α) or specialization (σ), since they determine the velocity of money (indeed, one can prove velocity is $2\alpha\sigma$ in the model). The model can also be extended to allow intrinsically heterogeneous types, or to assume that some agents sometimes cannot access the centralized market, which is interesting since either of these assumptions would make the distribution $F(m)$ nondegenerate, but in a manageable way. Hence, it should be possible to study the impact of policy on welfare via the distribution of real balances in a fairly simple setting (previously, this has required a fairly complicated environment, as in Molico [1999]).

It would also be interesting to consider pricing mechanisms other than bilateral bargaining, perhaps something along the lines of the competitive search equilibrium concept of Moen (1997) and Shimer (1995). It would also be useful to put capital and labor explicitly into the model. It turns out that one can do this rather easily. Indeed, we can write down a model that nests the framework in this paper and the standard neoclassical growth model. Equilibrium determines time paths for hours, capital, the consumption of general goods, and the consumption of special goods, $\{h, k_{+1}, c, q\}$. At least for the baseline version of the model, there is a dichotomy: the equilibrium conditions partition into a set determining $\{h, k_{+1}, c\}$, that look exactly like the standard (nonmonetary) growth model; and a single equation determining $\{q\}$ that looks exactly like the one in this paper (see Aruoba [2002]).

Universidad de SanAndrés

A Appendix

In this Appendix we first verify that the bargaining solutions are as claimed in the text. We then use these results to derive certain properties any equilibrium must satisfy. Finally, we use these properties to establish the existence of the value function V.

Lemma 1 *In a double coincidence meeting each agent produces q^* and no money changes hands.*

Proof. In double coincidence meetings, the symmetric Nash problem is

$$\max_{q_1, q_2, \Delta} \left[u\left(q_1\right) - c\left(q_2\right) - \phi\Delta\right]\left[u\left(q_2\right) - c\left(q_1\right) + \phi\Delta\right]$$

subject to $-m_2 \leq \Delta \leq m_1$, where q_1 and q_2 denote the quantities consumed by agents 1 and 2 and Δ is the amount of money 1 pays 2. The problem has a unique solution that is characterized by the first order conditions

$$\begin{aligned}
u'\left(q_1\right)\left[u\left(q_2\right) - c\left(q_1\right) + \phi\Delta\right] &= c'\left(q_1\right)\left[u\left(q_1\right) - c\left(q_2\right) - \phi\Delta\right] \\
c'\left(q_2\right)\left[u\left(q_2\right) - c\left(q_1\right) + \phi\Delta\right] &= u'\left(q_2\right)\left[u\left(q_1\right) - c\left(q_2\right) - \phi\Delta\right] \\
u\left(q_1\right) - u\left(q_2\right) + c\left(q_1\right) - c\left(q_2\right) - 2\phi\Delta &= \frac{(2/\phi)(\lambda_1 - \lambda_2)}{\{[u(q_1) - c(q_2) - \phi\Delta][u(q_2) - c(q_1) + \phi\Delta]\}^{-1/2}}
\end{aligned}$$

where λ_i is the multiplier on agent i's cash constraint. It is easy to see that $q_1 = q_2 = q^*$ and $\Delta = \lambda_1 = \lambda_2 = 0$ solves these conditions. ■

Lemma 2 *In a single coincidence meeting the bargaining solution is given by (7).*

Universidad de San Andrés

Proof. The necessary and sufficient conditions for (6) are

$$\theta\left[\phi d-c\left(q\right)\right]u'\left(q\right) = \left(1-\theta\right)\left[u\left(q\right)-\phi d\right]c'\left(q\right) \tag{29}$$

$$\theta\left[\phi d-c\left(q\right)\right]\phi = \left(1-\theta\right)\left[u\left(q\right)-\phi d\right]\phi -\lambda\left[u\left(q\right)-\phi d\right]^{1-\theta}\left[\phi d-c\left(q\right)\right]^{\theta} \tag{30}$$

where λ is the Lagrange multiplier on $d \leq m$. There are two possible cases: If the constraint does not bind, then $\lambda = 0$, $q = q^*$ and $d = m^*$. If the constraint binds then q is given by (29) with $d = m$, which is (9). ■

At this point we consider a non-recursive definition of equilibrium and deriving some of its properties. Given the bargaining solution $[q(m), d(m)]$, it suffices for our purpose to say that a monetary equilibrium is a path for $\{\phi_t, F_t\}$ with $\phi_t > 0$ such that, when individuals choose sequences $\{m_{t+1}\}$ the resulting sequence of distributions is $\{F_t\}$, and the money market clears in the sense that $\int mdF(m) = M$ at every date.

Universidad de
SanAndres

Lemma 3 *In any monetary equilibrium, $\beta\phi_{t+1} \leq \phi_t$ for all t. If $\theta \approx 1$ or u' is log concave then F is degenerate: $m_{t+1} = M$ with probability 1. In any equilibrium where F is degenerate, $\phi_t = G\left(\phi_{t+1}\right)$ for all t where G is a time-invariant continuous function.*

Proof. Given any path $\{\phi_t, F_t\}$ and m_0, the individual problem is

$$\max_{\{m_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[v\left(m_t, \phi_t, F_t\right) + \phi_t\left(m_t - m_{t+1}\right)\right]$$

where v is defined in (12) (which does not use the value function and is defined in terms date t variables only). We know v is C^{n-1}; thus, if a solution exists it satisfies the necessary conditions

$$\beta v_1 \left(m_{t+1}, \phi_{t+1}, F_{t+1}\right) + \beta \phi_{t+1} - \phi_t \leq 0, \ = 0 \text{ if } m_{t+1} > 0. \tag{31}$$

We have

$$v_1(m, \phi, F) = \begin{cases} \alpha\sigma \left[u'(q)\widehat{q}'(m) - \phi\right] & \text{if } m < m^* \\ 0 & \text{if } m \geq m^* \end{cases}$$

where $\widehat{q}'(m)$ is given in (10). If $\beta\phi_{t+1} > \phi_t$ then the left hand side of (31) is positive for $m > m^*$ and so the problem has no solution. Hence we must have $\beta\phi_{t+1} \leq \phi_t$ for all t.

In a monetary equilibrium, at least one agent must choose $m_{t+1} > 0$, and for this agent

$$\beta v_1 \left(m_{t+1}, \phi_{t+1}, F_{t+1}\right) + \beta \phi_{t+1} - \phi_t = 0. \tag{32}$$

As discussed in the text, a quick calculation verifies that if $\theta \approx 1$ or u' is log concave then $v_{11} < 0$, which implies (32) has a unique solution: all agents choose the same $m_{t+1} = M$. Hence F_{t+1} is degenerate in any monetary equilibrium. Finally, (32) implies $\phi_t = G\left(\phi_{t+1}\right)$, where G is continuous because v_1 is. ■

At this point the following issue arises: although we know that in any equilibrium $\phi_t = G\left(\phi_{t+1}\right)$, to use dynamic programming we need to know $\phi_{t+1} = \Phi\left(\phi_t\right)$. But G may not be invertible. Our strategy is to restrict attention to equilibria where $\phi_{t+1} = \Phi\left(\phi_t\right)$ and Φ is continuous. Obviously this includes all steady state equilibria, and it includes all possible equilibria in the case where G is invertible, but it also includes many other interesting

Universidad de San Andrés

cases, such as the cycles constructed in the text. All that it rules out are the following somewhat exotic possibilities. First note that any equilibrium involves selecting an initial price ϕ_0, or equivalently q_0 since we can always invert $\phi_0 = \phi(q_0)$ by (9), and then selecting future values from the correspondence $\phi_{t+1} = G^{-1}(\phi_t)$. We simply insist that the selection ϕ_{t+1} from $G^{-1}(\phi_t)$ cannot vary with time or the value of ϕ_t.

That is to say, while the value ϕ_{t+1} obviously varies with ϕ_t, the *rule for choosing which branch* of G^{-1} from which to select ϕ_{t+1} must be constant. We know that this is possible for a large class of dynamic equilibria; e.g., one can use the rule "always select the lowest branch of G^{-1}" and construct equilibria where $\phi_t \to 0$ from any initial ϕ_0 in some interval $(0, \bar{\phi}_0)$. While we may not pick up all possible equilibria due to this restriction, we pick up a lot. And we emphasize that the purpose of this restriction is limited: we already know that $\beta\phi_{t+1} \leq \phi_t$ for all t and that F is degenerate in any equilibrium; all we are doing here is trying to guarantee $\phi_{t+1} = \Phi(\phi_t)$ where Φ is continuous in order to prove the existence of the value function V and use dynamic programming.

Given $\phi_{t+1} = \Phi(\phi_t)$ where Φ is continuous, we still need to bound ϕ. We do this with M constant, but the arguments are basically the same when M is varying over time as long as we work with real balances $z = \phi M$.

Lemma 4 *Assume* $\sup U(X) > \overline{V} \equiv \frac{u(q^*)+U(X^*)}{1-\beta}$. *Then in any equilibrium* ϕ *is bounded above by* $\overline{\phi} = \overline{z}/M$, *where* $U(\overline{z}) = \overline{V}$.

Proof. Clearly lifetime utility V in any equilibrium is bounded by $\bar{V}$. Consider a candidate equilibrium with $\phi M > \bar{z}$ at some date. In the candidate equilibrium, an individual with $m = M$ would want to deviate by trading all his money for general goods since $U(\phi M) > \overline{V}$. Hence, ϕM is bounded above by $\bar{z}$. ■

Next, we verify the existence and uniqueness of the value function.

Lemma 5 *Let $S = \mathbb{R}\times [0,\overline{\phi}]$ with $\overline{\phi}$ defined as in Lemma 4, and consider the metric space given by $\mathbb{C} = \{\hat{v} : S \to \mathbb{R} \mid \hat{v}$ is bounded and continuous$\}$ together with the sup norm, $\|\hat{v}\| = \sup |\hat{v}(m,\phi)|$. Define*

$$\mathbb{C}' = \left\{\hat{V} : S \to \mathbb{R} | \hat{V}(m,\phi) = \hat{v}(m,\phi) + \phi m \text{ for some } \hat{v} \in \mathbb{C}\right\}.$$

Let $\Phi : [0,\overline{\phi}] \to [0,\overline{\phi}]$ be a continuous function, and define the operator $T : \mathbb{C}' \to \mathbb{C}'$ by

$$\left(T\hat{V}\right)(m,\phi) = \sup_{m_{+1}} \left\{ v(m,\phi) + \phi m - \phi m_{+1} + \beta \hat{V}[m_{+1}, \Phi(\phi)] \right\}$$

where $v(m,\phi)$ is defined in (12). Then T has a unique fixed point $V \in \mathbb{C}'$.

Proof. First we show $T : \mathbb{C}' \to \mathbb{C}'$. For every $\hat{V} \in \mathbb{C}'$ we can write

$$\left(T\hat{V}\right)(m,\phi) = v(m,\phi) + \phi m + \sup_{m_{+1}} w[m_{+1}, \Phi(\phi)]$$

where $w[m_{+1}, \Phi(\phi)] = \beta\hat{v}[m_{+1}, \Phi(\phi)] + \beta\phi m_{+1} - \phi m_{+1}$ for some $\hat{v} \in \mathbb{C}$. Since $\hat{v}$ is bounded, there exists a $\overline{m}$ such that $\beta w[0, \Phi(\phi)] > \beta w[m_{+1}, \Phi(\phi)]$ for all $m_{+1} \geq \overline{m}$. Therefore,

$$\sup_{m_{+1}} w[m_{+1}, \Phi(\phi)] = \max_{m_{+1} \in [0,\overline{m}]} w[m_{+1}, \Phi(\phi)],$$

and the maximum is attained. Using $w^*(\phi)$ to denote the solution, we have $T\hat{V}(m,\phi) = v(m,\phi) + w^*(\phi) + \phi m \in \mathbb{C}'$, since $w^*(\phi) \in \mathbb{C}$ by the Theorem of the Maximum and $v(x,\phi) \in \mathbb{C}$ from the bargaining solution.

We now show T is a contraction mapping. Define the norm $\left\|\hat{V}_1 - \hat{V}_2\right\| = \sup|\hat{v}_1(m,\phi) - \hat{v}_2(m,\phi)|$ and consider the metric space $(\mathbb{C}', \|\cdot\|)$. Fix $(m,\phi) \in S$. Then, letting $m^i_{+1} = \underset{m_{+1}\in[0,\overline{m}]}{\arg\max}\left\{\beta\hat{V}_i[m_{+1}, \Phi(\phi)] - \phi m_{+1}\right\}$ we have

$$\begin{aligned} T\hat{V}_1 - T\hat{V}_2 &= \left\{\beta\hat{V}_1\left[m^1_{+1}, \Phi(\phi)\right] - \phi m^1_{+1}\right\} - \left\{\beta\hat{V}_2\left[m^2_{+1}, \Phi(\phi)\right] - \phi m^2_{+1}\right\} \\ &\leq \beta\left|\hat{V}_1\left[m^1_{+1}, \Phi(\phi)\right] - \hat{V}_2\left[m^1_{+1}, \Phi(\phi)\right]\right| \leq \beta\left\|\hat{V}_1 - \hat{V}_2\right\|. \end{aligned}$$

Similarly, we can derive $T\hat{V}_2 - T\hat{V}_1 \leq \beta\left\|\hat{V}_1 - \hat{V}_2\right\|$. Hence $\left|T\hat{V}_2 - T\hat{V}_1\right| \leq \beta\left\|\hat{V}_1 - \hat{V}_2\right\|$. Taking the supremum over (m,ϕ), $\left\|T\hat{V}_1 - T\hat{V}_2\right\| \leq \beta\left\|\hat{V}_1 - \hat{V}_2\right\|$ and T satisfies the definition of a contraction.

We now argue that $(\mathbb{C}', \rho)$ is complete. Clearly, if $\hat{V}_n(m,\phi) = \hat{v}_n(m,\phi) + \phi m$ is a Cauchy sequence in $\mathbb{C}'$, then $\{\hat{v}_n(m,\phi)\}$ is a Cauchy sequence in $\mathbb{C}$. Since $(\mathbb{C}, \|\cdot\|)$ is complete (Theorem 3.1 in Stokey and Lucas [1989]), $\hat{v}_n \to v \in \mathbb{C}$. Set $V = v + \phi m$ and it is immediate that $\hat{V}_n \to V \in \mathbb{C}'$. Therefore $(\mathbb{C}', \rho)$ is complete. It now follows from the Contraction Mapping Theorem (Theorem 3.2 in Stokey and Lucas [1989]) that T has a unique fixed point $V \in \mathbb{C}'$. ■

QUAERERE VERUM
Universidad de
SanAndrés

References

[1] Aiyagari, S. Rao, and Neil Wallace. "Existence of Steady States with Positive Consumption in the Kiyotaki-Wright Model." *Review of Economic Studies* 58(5) (October 1991): 901-916.

[2] Aiyagari, S. Rao, Neil Wallace and Randall Wright. "Coexistence of Money and Interest-Bearing Securities," *Journal of Monetary Economics* 37 (1996): 397-420.

[3] Alvarez, Fernando, and Nancy L. Stokey. "Dynamic Programming with Homogeneous Functions." *Journal of Economic Theory* 82 (1998): 167-189.

[4] Aruoba, S. Boragan. "Search, Money, and Capital: A Neoclassical Dichotomy." unpublished manuscript, 2002.

[5] Azariadis, Costas. *Intertemporal Macroeconomics.* Oxford: Blackwell Publishers, 1993.

[6] Berentsen, Aleksander. "Money Inventories in Search Equilibrium." unpublished manuscript, 1999.

[7] Berentsen, Aleksander, and Guillaume Rocheteau. "The Role of Money in Double Coincidence Environments." unpublished manuscript, 2000a.

[8] Berentsen, Aleksander, and Guillaume Rocheteau. "On the Efficiency of Monetary Exchange." unpublished manuscript, 2000b.

Universidad de SanAndrés

[9] Berentsen, Aleksander, Guillaume Rocheteau and Shouyong Shi. "Friedman Meets Hosios: Efficiency in Search Models of Money." unpublished manuscript, 2001.

[10] Burdett, Kenneth. "A Useful Restriction on the Offer Distribution in Job Search Models," in G. Eliasson, B. Holmlund and F. P. Stafford, eds., *Studies in Labor Market Behavior: Sweden and the United States.* Stockholm, Sweden: I.U.I. Conference Report, 1981.

[11] Burdett, Kenneth, Alberto Trejos and Randall Wright. "Cigarette Money." *Journal of Economic Theory* 99 (2001): 117-42.

[12] Camera, Gabriele, and Dean Corbae. "Money and Price Dispersion." *International Economic Review* 40(4) (November 1999): 985-1008.

[13] Cavalcanti, Ricardo and Neil Wallace. "A Model of Private Bank-Note Issue," *Review of Economic Dynamics* 2 (1999): 104-136.

[14] Cavalcanti, Ricardo, and Neil Wallace. "Inside and Outside Money as Alternative Media of Exchange," *Journal of Money, Credit and Banking* 31 (1999): 443-457

[15] Christiano, Lawrence, Martin Eichenbaum and Charles Evans. "Sticky Price and Limited Participation Models: A Comparison," *European Economic Review* 41 (June 1997): 1173-1200.

QUAERERE VERUM
Universidad de
SanAndrés

[16] Coles, Melvyn, and Randall Wright. "A Dynamic Equilibrium Model of Search, Bargaining and Money." *Journal of Economic Theory* 78(1) (January 1998): 32-54.

[17] Cooley, Thomas and Gary Hansen. "The Inflation Tax in a Real Business Cycle Model." *American Economic Review* 79 (September 1989): 733-748.

[18] Duffie, Darrell, John Geanakoplos, Andreu Mas-Colell and Andrew McLennan. "Stationary Markov Equilibria." *Econometrica* 62 (1994): 745-782.

[19] Faig, Miguel. "A Search Theory of Money with Commerce and Neoclassical Production." unpublished manuscript, 2001.

[20] Green, Edward, and Ruilin Zhou. "A Rudimentary Model of Search with Divisible Money and Prices." *Journal of Economic Theory* 81(2) (August 1998): 252-271.

[21] Head, Allen and Shouyong Shi. "A Fundamental Theory of Exchange Rates and Direct Currency Trades." unpublished manuscript, 2000.

[22] Head, Allen and Alok Kumar. "Price-posting, price dispersion, and inflation in a random matching model." unpublished manuscript, 2001.

[23] Hosios, Arthur J. "On the Efficiency of Matching and Related Models of Search and Unemployment." *Review of Economic Studies* 57 (April 1990): 279-298.

[24] Kiyotaki, Nobuhiro and Randall Wright. "On Money as a Medium of Exchange." *Journal of Political Economy* 97(4) (August 1989): 927-954.

[25] Kiyotaki, Nobuhiro and Randall Wright. "A Contribution to the Pure Theory of Money." *Journal of Economic Theory* 53(2) (April 1991): 215-235.

[26] Kiyotaki, Nobuhiro and Randall Wright. "A Search-Theoretic Approach to Monetary Economics." *American Economic Review* 83 (3) (March 1993): 63-77.

[27] Kocherlakota, Narayana. "Money is Memory." *Journal of Economic Theory* 81(2) (August 1998): 232-251.

[28] Li, Victor. "The Optimal Taxation of Fiat Money in Search Equilibrium," *International Economic Review* 36(4) (November 1995): 927-42.

[29] Li, Yiting and Randall Wright. "Government Transaction Policy, Media of Exchange, and Prices." *Journal of Economic Theory* 81(2) (August 1998): 290-313.

[30] Lucas, Robert E., Jr. "Equilibrium in a Pure Currency Economy." In *Models of Monetary Economies*, edited by John H. Kareke and Neil Wallace. Minneapolis: Federal Reserve Bank of Minneapolis.

[31] Matsuyama, Kiminori, Nobuhiro Kiyotaki and Ahihiro Matsui. "Toward a Theory of International Currency." *Review of Economic Studies* 60(2) (April 1993): 283-307.

Universidad de SanAndrés

[32] Moen, Espen R. "Competitive Search Equilibrium," *Journal of Political Economy* 105 (1997): 385-411.

[33] Molico, Miguel. "The Distribution of Money and Prices in Search Equilibrium." unpublished manuscript, 1999.

[34] Rauch, Bernard. "A Divisible Search Model of Fiat Money: A Comment." *Econometrica* 68(1) (January 2000): 149-156.

[35] Shi, Shouyong. "Money and Prices: A Model of Search and Bargaining." *Journal of Economic Theory* 67(2) (December 1995): 467-496.

[36] Shi, Shouyong. "A Divisible Search Model of Fiat Money." *Econometrica* 64(1) (January 1997): 75-102.

[37] Shi, Shouyong. "Search for a Monetary Propagation Mechanism." *Journal of Economic Theory* 81(2) (August 1998): 314-352.

[38] Shi, Shouyong. "Search, Inflation and Capital Accumulation." *Journal of Monetary Economics* 44(1) (August 1999): 81-103.

[39] Shimer, Robert. "Contracts in Frictional Labor Markets," mimeo, Princeton University, 1995.

[40] Stokey, Nancy and Lucas, Robert E. Jr. *Recursive Methods in Economic Dynamics.* Cambridge: Harvard University Press, 1989.

[41] Taber, Alexander, and Neil Wallace. "A Matching Model with Bounded Holdings of Indivisible Money." *International Economic Review* 40(4) (November 1999): 961-984.

QUAERERE VERUM
Universidad de
SanAndrés

[42] Trejos, Alberto, and Randall Wright. "Search, Bargaining, Money and Prices." *Journal of Political Economy* 103(1) (February 1995): 118-141.

[43] Velde, Francois, Warren Weber and Randall Wright. "A Model of Commodity Money, with Applications to Gresham's Law and the Debasement Puzzle." *Review of Economic Dynamics* 2 (1999): 291-323.

[44] Walsh, Carl E. *Monetary Theory and Policy.* MIT Press, 1998.

[45] Wallace Neil. "Short Run and Long Run Effects of Changes in Money in a Random Matching Model." *Journal of Polictical Economy* 105 (2000): 1293-1307.

[46] Wallace, Neil. "Knowledge of Individual Histories and Optimal Payment Arrangements." Federal Reserve Bank of Minneapolis Quarterly Review, 24 (3) (Summer 2000): 11-21.

[47] Wallace, Neil. "Whither Monetary Economics?" *International Economic Review* 42 (November 2001): 847-870.

[48] Williamson, Steve, and Randall Wright. "Barter and Monetary Exchange under Private Information." *American Economic Review* 84(1) (March 1994): 104-123.

[49] Zhou, Ruilin. "Currency Exchange in a Random Matching Model. " *Review of Economic Studies* 64(2) (April 1997): 289-310.

Universidad de SanAndres

[50] Zhou, Ruilin. "Individual and Aggregate Real Balances in a Random-Matching Model." *International Economic Review* 40(4) (November 1999): 1009-1038.

QUAERERE VERUM
Universidad de
SanAndrés