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"Inferencia bayesiana: campos de coincidencia de opiniones."

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DEFINETTIAN CONSENSUS

ABSTRACT. It is always possible to construct a real function ϕ, given random quantities X and Y with continuous distribution functions F and G, respectively, in such a way that $\phi(X)$ and $\phi(Y)$, also random quantities, have both the same distribution function, say H. This result of De Finetti introduces an alternative way to somehow describe the "opinion" of a group of experts about a continuous random quantity by the construction of Fields of coincidence of opinions (FCO). A Field of coincidence of opinions is a finite union of intervals where the opinions of the experts coincide with respect to that quantity of interest. We speculate on (dis)advantages of Fields of Opinion compared to usual "probability" measures of a group and on their relation with a continuous version of the well-known Allais' paradox.

1. INTRODUCTION

The main object of this paper is to review a result about transformations of continuous random quantities presented by De Finetti in (1953) and the applications to group decision-making. We will introduce the problem solved by De Finetti by first recalling a well-known theorem about transformations of random quantities.

THEOREM 1. *Let X be a real random variable with continuous distribution function F and H be any other distribution function. There is a real transformation f such that $Z = f(X)$ has distribution function H.*

Theorem 1 applies to a single random variable in the sense that if X and Y are random variables with continuous distribution functions, then $f(X)$ and $f(Y)$ will not have necessarily the same distribution. In other words, given two continuous random variables X and Y, it is always possible to construct two new random variables $Z = f(X)$ and $W = g(Y)$, both having a given distribution function

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H. However, in general, f and g may differ. In this context, a question arises: Is there a real function ϕ such that the random variables $\phi(X)$ and $\phi(Y)$ both have the same distribution function H ?

This question is affirmatively answered by De Finetti in his 1953 paper. De Finetti shows a real function ϕ satisfying the conditions stated in the question above and outlines the construction of the random variables $\phi(X)$ and $\phi(Y)$ and the derivation of their distribution functions.

We will present, with full details, the theorem of the existence of such real functions ϕ and will discuss an interpretation of this result.

2. MAIN RESULT

We first state the theorem mentioned in the previous section as well as the basic argument for its proof. We next construct a real function φ which figures in the demonstration of the main result. Finally, we define the random variables $\varphi(X)$ and $\varphi(Y)$ and determine their distribution functions. (We are keeping throughout De Finetti's original definition for distribution functions, left-continuous.)

Initially, let us state the main theorem:

THEOREM 2. *(Bruno de Finetti). Let X and Y be random variables with continuous distribution functions F and G, respectively, and H be any other distribution function. There is a real function ϕ such that the random variables $\phi(X)$ and $\phi(Y)$ both have the same distribution function H.*

Proof outline: It is sufficient to prove the existence of a real function φ such that the random variables $\varphi(X)$ and $\varphi(Y)$ have common uniform distribution on (0, 1), since if $\phi = H^{-1} \circ \varphi$, $\phi(X)$ and $\phi(Y)$ will both have distribution function H. Thus, we proceed to construct the uniformly distributed random variables $\varphi(X)$ and $\varphi(Y)$, without loss of generality.

2.1. *Construction of the Function φ*

The construction of the aforementioned real function φ is based on two properties of continuous distribution functions described in the sequel.

We first note that since to every point $(x, y) \in \mathbb{R}^2$, with $x < y$, corresponds a unique interval $(x, y]$ of real numbers, $C_X = \{(a, b) \in \mathbb{R}^2 : F(b) - F(a) = \frac{1}{2}\ ,\ -\infty < a < b < +\infty\}$ may be seen as the set of all intervals of real numbers having X-probability $\frac{1}{2}$. We establish the following propositions:

PROPOSITION 1. *Let X and Y be random variables with continuous distribution functions F and G, respectively. There is an interval of real numbers $I_1 = (a, b]$, with $-\infty \leqslant a < b < +\infty$, satisfying*

$$F(b) - F(a) = G(b) - G(a) = \frac{1}{2}.$$

Proof: If F and G have a common median, the conclusion is immediate. On the other hand, if F and G have no common median, we will prove the existence of a point $(a, b) \in C_X$ such that $G(b) - G(a) = \frac{1}{2}$. For this purpose, we define the function $D : C_X \to \mathbb{R}$ by:

$$D(x, y) = G(y) - G(x).$$

D is obviously continuous on its domain C_X. C_X is connected (see Esteves [1997]) and, since D is continuous, it follows that the image of D is also connected and, in particular, is an interval of real numbers. But there are points (a_1, b_1) and $(a_2, b_2) \in C_X$ such that $D(a_1, b_1) < \frac{1}{2}$ and $D(a_2, b_2) > \frac{1}{2}$ (see Esteves, 1997). The image of the function D is therefore an interval containing a value smaller than $\frac{1}{2}$ and another greater than $\frac{1}{2}$. Thus, there is an interval $(a, b) \in C_X$ such that

$$D(a, b) = G(b) - G(a) = \frac{1}{2} = F(b) - F(a)\ \text{, since } (a, b) \in C_X.$$

We emphasize that in situations where there are more than one interval satisfying Proposition 1, we will denote by I_1 the interval having the lowest infimum among those satisfying this result, in order to avoid any ambiguity (we here admit, by a misuse of notation, $-\infty$ as the infimum of an unbounded interval). This choice having been made and still existing more than one interval satisfying proposition 1, I_1 will represent the interval with lowest supremum. Furthermore, we will always consider I_1 closed at the right and open at the left and will denote by I_0 the complementary set of I_1 relatively to $\mathbb{R}$.

We now state another property of continuous distribution functions.

PROPOSITION 2. *Let us assume the conditions of Proposition 1 and the sets I_0 and I_1 derived from it. Then:*
(i) there is a set $I_{01} \subset I_0$ such that $P(X \in I_{01}) = P(Y \in I_{01}) = \frac{1}{4}$ and
(ii) there is a set $I_{11} \subset I_1$ such that $P(X \in I_{11}) = P(Y \in I_{11}) = \frac{1}{4}$.

Proof:

(i) Let $I_1 = (a, b]$.

Let us suppose $-\infty < a < b < +\infty$. By Proposition 1, $F(b) - F(a) = G(b) - G(a) = \frac{1}{2}$. Let us define the following distribution function $\overline{F}$ derived from F

$$\overline{F}(x) = \begin{cases} 2F(x) & \text{for } x < a \\ 2\{F(x+b-a) - \frac{1}{2}\} & \text{for } x \geqslant a. \end{cases}$$

Analogously, let us define $\overline{G}$ by:

$$\overline{G}(x) = \begin{cases} 2G(x) & \text{for } x < a \\ 2\{G(x+b-a) - \frac{1}{2}\} & \text{for } x \geqslant a. \end{cases}$$

By Proposition 1, $\exists\ (a_0, b_0] \subset \mathbb{R}$ such that $\overline{F}(b_0) - \overline{F}(a_0) = \overline{G}(b_0) - \overline{G}(a_0) = \frac{1}{2}$.

Suppose $b_0 < a$. In this case, $\overline{F}(b_0) - \overline{F}(a_0) = 2F(b_0) - 2F(a_0)$ and $\overline{G}(b_0) - \overline{G}(a_0) = 2G(b_0) - 2G(a_0)$. Then,

$$\begin{aligned} &\overline{F}(b_0) - \overline{F}(a_0) = \overline{G}(b_0) - \overline{G}(a_0) = \tfrac{1}{2} \\ &\Rightarrow 2F(b_0) - 2F(a_0) = 2G(b_0) - 2G(a_0) = \tfrac{1}{2} \\ &\Rightarrow F(b_0) - F(a_0) = G(b_0) - G(a_0) = \tfrac{1}{4} \end{aligned}$$

and the result in (i) is proved.

Analogously, we can prove the other cases, including the situation $a = -\infty$ and part (ii).

Similarly to what was established for the interval I_1, in the situation where we have more than one subset of I_1 satisfying Proposition 2, part (ii), we will consider the interval with lowest extremes as I_{11}. We also will denote by I_{01} the subset of I_0 satisfying part (i) of Proposition 1.2 formed by the smallest number of intervals. Again, if there is more than one subset of I_0 in these conditions, we will denote by I_{01} the one with lowest infimum. Finally, the complementary set of I_{11} relatively to I_1 will be denoted by I_{10} and the complementary set of I_{01} relatively to I_0 will be denoted by I_{00}.

In general, proceeding successively in this way, we can obtain, $\forall n \in \mathbb{N}$, 2^n disjoint sets $I_{i_1 \ldots i_n}$, such that, $\forall (i_1, \ldots, i_n) \in \{0, 1\}^n$

$$P(X \in I_{i_1 \ldots i_n}) = P(Y \in I_{i_1 \ldots i_n}) = \left(\frac{1}{2}\right)^n \text{ with}$$

$$I_{i_1 \ldots i_n} = I_{i_1 \ldots i_n, 0} \bigcup I_{i_1 \ldots i_n, 1}, \quad \forall n \in \mathbb{N}.$$

Stated propositions 1 and 2 and their extensions yield the real function φ which makes the random variables $\varphi(X)$ and $\varphi(Y)$ uniformly distributed over (0, 1).

We define $\varphi : \mathbb{R} \to [0, 1]$ by:

$$\varphi(x) = \sum_{n=1}^{\infty} i_n \left(\frac{1}{2}\right)^n,$$

where $i_1, i_2, \ldots$ are such that $x \in I_{i_1 \ldots i_n}$, $\forall n \geqslant 1$ (here, the sets $I_{i_1 \ldots i_n}$ correspond to the sets constructed via the distribution functions F and G). It should then be noted that φ is also a function of the distribution functions F and G, but we will omit these arguments when referring to the function φ in order to keep the notation easy.

Analysing the expression of φ, we see that the function associates to each real number x the element of the interval [0, 1] having a dyadic representation (expansion) given by $0, i_1 i_2 \ldots$, with $x \in I_{i_1 \ldots i_n}$, $\forall n \geqslant 1$. It can be proved in a straightforward manner that φ is well-defined.

2.2. *Determining the Distribution of* φ

Let us now prove that $\varphi(X)$ and $\varphi(Y)$ both have uniform distribution on the interval (0, 1). We will consider the distribution function of $\varphi(X)$,

$$\begin{aligned} F_{\varphi(X)}(t) &= P(\varphi(X) < t) = P(X \in \varphi^{-1}((-\infty, t)) \\ &= P_X(\{x \in \mathbb{R} : \varphi(x) < t\}), \end{aligned}$$

where P_X is the probability measure on $(\mathbb{R}, \mathcal{B})$ induced by the random variable X and $\varphi^{-1}(A)$ is the inverse image of the set $A \in \mathcal{B}$ by the function φ.

Let us consider $0 < t \leqslant 1$; fixing $t \in (0, 1]$, this point t may be written as $t = \sum_{n=1}^{\infty} d_n(t)(\frac{1}{2})^n$, where $d_1(t), d_2(t)...$ are such that $0, d_1(t)d_2(t)...$ is a dyadic expansion of t. In order to avoid any ambiguity in the definition of the dyadic expansion of a real number $t \in (0, 1]$, we will here consider for t the infinite dyadic representation, that is, the representation having an infinite number of $1's$ (for instance, for $t = \frac{1}{2}$, we will consider the expansion 0, 01111... instead of the expansion 0, 10000...). We then have:

$$\{x \in \mathbb{R} : \varphi(x) < t\} = \bigcup_{n=1}^{\infty} A_n,$$

where $\{A_n : n \geqslant 1\}$ is the sequence of sets defined by

$$A_1 = \begin{cases} \emptyset & \text{if } d_1(t) = 0 \\ I_0 & \text{if } d_1(t) = 1 \end{cases}$$

and

$$A_n = \begin{cases} \emptyset & \text{if } d_n(t) = 0 \\ I_{d_1(t)...d_{n-1}(t), 1-d_n(t)} & \text{if } d_n(t) = 1 \end{cases}, \quad n > 1.$$

We then obtain, by taking $x_0 \in \{x \in \mathbb{R} : \varphi(x) < t\}$ and considering $\varphi(x_0) = \sum_{n=1}^{\infty} i_n \left(\frac{1}{2}\right)^n$,

$$x_0 \in \{x \in \mathbb{R} : \varphi(x) < t\} \Leftrightarrow \varphi(x_0) < t \Leftrightarrow \sum_{n=1}^{\infty} i_n \left(\frac{1}{2}\right)^n < \sum_{n=1}^{\infty} d_n(t) \left(\frac{1}{2}\right)^n.$$

Since we are considering an infinite dyadic expansion for t, it follows that the last inequality above is true if, and only if,

$$\exists\, n_0 \in \mathbb{N} \text{ such that } n_0 = \inf\{n \in \mathbb{N} : i_n \neq d_n(t) \text{ and } i_n = 1 - d_n(t) = 0\} \Leftrightarrow$$

$$\Leftrightarrow \exists\, n_0 \in \mathbb{N} \text{ such that } x_0 \in I_{d_1(t)...d_{n_0-1}(t), 0} \text{ and } d_{n_0}(t) = 1 \Leftrightarrow x_0 \in \bigcup_{n=1}^{\infty} A_n.$$

Therefore,

$$\{x \in \mathbb{R} : \varphi(x) < t\} = \bigcup_{n=1}^{\infty} A_n.$$

However, for every $n \in \mathbb{N}$, each set of the form $I_{i_1 \ldots i_n}$ is formed by a finite union of intervals (at most $n + 1$ intervals; see Esteves, 1997), and since $A_n = \emptyset$ or A_n is of the form $I_{i_1 \ldots i_n}$, it follows that $A_n \in \mathcal{B}, \forall n \geqslant 1$ and $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$.

Finally, let us determine the distribution function of $\varphi(X)$,

$$F_{\varphi(X)}(t) = P_X(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P_X(A_n),$$

as $\{A_n : n \geqslant 1\}$ is a sequence of pairwise disjoint sets.

However, if $d_n(t) = 0$, then $A_n = \emptyset$ and $P_X(A_n) = 0$. If $d_n(t) = 1$, then $P_X(A_n) = P_X(I_{d_1(t) \ldots d_{n-1}(t), 1-d_n(t)}) = (\frac{1}{2})^n$, as, by construction, the set $I_{d_1(t) \ldots d_{n-1}(t), 1-d_n(t)}$ contains $(\frac{1}{2})^n$ of the distribution of X. In this way:

$$P_X(A_n) = \begin{cases} 0 & \text{if } d_n(t) = 0 \\ (\frac{1}{2})^n & \text{if } d_n(t) = 1 \end{cases} \quad \text{or} \quad P_X(A_n) = \left(\frac{1}{2}\right)^n d_n(t).$$

Recalling the expression of the distribution function of $\varphi(X)$, we have

$$F_{\varphi(X)}(t) = \sum_{n=1}^{\infty} d_n(t) \left(\frac{1}{2}\right)^n = t,$$

since $0, d_1(t) d_2(t) \ldots$ is a dyadic expansion of t. We then obtain that $\varphi(X)$ is a random variable uniformly distributed on $(0, 1)$. The proof that $\varphi(Y) \sim \mathcal{U}(0, 1)$ is analogous.

It is interesting to emphasize that the conditions of Theorem 2 do not mention the probability spaces where X and Y are defined; references are made only to the structure of the distribution functions F e G, so that Theorem 2 is valid also for random variables defined in distinct probability spaces. We also note that X and Y need neither to be absolutely continuous random variables nor to possess moments.

This interesting result has not been widely known. As far as we know, the only reference to it was made by Cifarelli and Regazzini (1996) in a paper where De Finetti's works are catalogued. They comment that for the paper it is "difficult to find suitable pigeon holes" in their classification. We are not aware of any translation of the paper, particularly to English. De Finetti preferred to write his early papers in French or Italian and, according to Barlow (1992), this could be a reason for the relative little attention his work got in the English-speaking world not to mention the non-subjectivistic scenario of the Fifties.

In the next section, we present an interpretation of the result just proved.

3. INTERPRETATION

The most interesting focus for Theorem 2, according to De Finetti, corresponds to the situation in which the random variables X and Y are related to a unique random quantity of interest, instead of two distinct quantities of interest, and F and G are the opinions of two individuals about that unique random quantity. This formulation is, of course, natural from the subjectivistic standpoint.

In this context, when two persons express their opinions about a given random quantity of interest, we have the following fact derived from the construction of the real function φ (and of the random variables $\varphi(X)$ and $\varphi(Y)$): it is always possible to construct a finite union of intervals, which De Finetti named Fields of Coincidence of Opinions (FCO), in which the opinions of both individuals about the quantity of interest coincide. In other words, it is always possible to construct a finite union of intervals that contains, for any level $\alpha \in (0, 1)$, at least $1 - \alpha$ of the distributions of X and Y simultaneously. We state this fact formally by the following proposition:

PROPOSITION 3. *Let X and Y be random variables with continuous distribution functions F and G, respectively. Then, $\forall\alpha \in (0, 1)$, there is a finite union of intervals $B = B(\alpha)$ such that*

$$P_X(B) = P_Y(B) \geqslant 1 - \alpha.$$

Proof: Let us fix $\alpha \in (0, 1)$. We know that there is a natural number $n_0 = n_0(\alpha) \in \mathbb{N}$ such that $1 - (\frac{1}{2})^{n_0} \geqslant 1 - \alpha$. We then need only to define a sequence of sets $\{B_n : n \geqslant 1\}$ by

$$B_1 = I_1, \quad B_2 = I_{01}, \ldots, \quad B_n = I_{\underbrace{0\ldots01}_{n-1 \text{ zeroes}}}, \ldots$$

and take $B = \bigcup_{n=1}^{n_0} B_n$..We then have

$$P_X(B) = P_X\left(\bigcup_{n=1}^{n_0} B_n\right) = \sum_{n=1}^{n_0} P_X(B_n),$$

where the last equality follows from the fact that $\{B_n : n \geqslant 1\}$ is a sequence of pairwise disjoint sets. And since, by construction, $P_X(B_n) = (\frac{1}{2})^n$, $\forall n \geqslant 1$, it follows that

$$P_X(B) = \sum_{n=1}^{n_0}\left(\frac{1}{2}\right)^n = 1 - \left(\frac{1}{2}\right)^{n_0} \geqslant 1 - \alpha.$$

In an analogous way, we verify that $P_Y(B) = 1 - (\frac{1}{2})^{n_0} \geqslant 1 - \alpha$, concluding the demonstration of Proposition 3.

The result of Proposition 3 can be extended to any finite number of continuous distribution functions. Thus, it is possible to establish fields of coincidence of opinions for a finite group of individuals. This fact, according to De Finetti, hints a possibility of characterizing the "conjoint opinion" of a group of experts, as discussed below.

Let us consider the situation where a group of experts has to make a decision jointly and, for this purpose, they have to tell their opinions about a certain random quantity of interest via the elicitation of their respective distribution functions (that we suppose continuous). The construction of the fields of coincidence of opinions sketches an alternative method of expressing what the joint "opinion" of these experts would look like, as some properties of fields of coincidence of opinions are requirements to characterize the "opinion" of a group of experts.

At first, we emphasize that the fields of coincidence can be seen as a genuine attribution of probability from the group, differently of the usual procedures of combining probabilities, which produce probability distributions having no meaning in the subjectivistic approach. In other words, the probability distributions resulting from the processes of mixture do not correspond to the opinion of anybody, opposing De Finetti's viewpoint.

Another positive point of this method is that all individual opinions are preserved in the construction of the fields of coincidence of opinions. This property gives to this method an objective character in the sense that individual opinions are preserved and there is no need for any member to give up or compromise his belief about the random quantity. At this point it may be interesting to recall De Finetti's notion of "objectivity". $X = x$ is "objective" if there is unanimity about x among the members of the group (see Wechsler, 1993; Dawid, 1982).

On the other hand, the method based on fields of coincidence also presents some limitations. Initially, we observe that a field of coincidence of opinions does not determine a probability distribution as it consists only of a finite union of intervals and their respective uncertainty rates according to all members of the group. Thus, the normative Bayesian theory for decision-making (based on expected utility maximization) does not apply to any procedure of decision-making based on fields of coincidence as a description of the "opinion" of the group. It is open to speculation whether it is possible to maximize "expected" common utility (with respect to a field of coincidence).

Another deficiency that we can point out in this method is the absence of an axiomatic support, based on coherence, which would justify the adoption of fields of coincidence as a representation of the "opinion" of a group of experts. This disadvantage arises because there is no general concept of joint coherence (or rationality). Notwithstanding Arrow's (1951) impossibility result, much of current research in group decision theory has been devoted to establish such a concept and, consequently, a numerical transcription of the uncertainty of a group of experts (see Nau, 1992 and 1995 for definitions of joint coherence).

In this context, where there is no normative theory for decision-making, but many attempts to characterize joint coherence, we think that the existence of the fields of coincidence of opinions may contribute to the discussion on this question. This discussion turns out to be even broader as there is " a point which is becoming increasingly better understood in group decision theory, namely that a group of Bayesians cannot always be fully Bayesian even when its members would want it to be" (Genest and Zidek, 1986).

We now show some examples of fields of coincidence of opinions.

EXAMPLE 1. Let X and Y be random variables uniformly distributed on $(1, 2)$ and $(2, 3)$, respectively.

A field of coincidence of opinions with $\alpha = 0{,}125$ would be the union of intervals $(1, \frac{11}{8}] \bigcup (\frac{3}{2}, \frac{23}{8}]$ (corresponding to $I_1 \bigcup I_{01} \bigcup I_{001}$, as the construction in section 2.1), or the interval $(\frac{9}{8}, \frac{23}{8}]$. As the supports of the distributions of X and Y are disjoint, no field of coincidence will be contained in the intersection of these supports.

EXAMPLE 2. Let X and Y be random variables normally distributed with common mean 0 and variances 1 and 4, respectively.

Here, a field of coincidence of opinions with $\alpha = 0{,}25$ is the set $\mathbb{R}^- \bigcup (0,635\ ;\ 2,35]$. In this case, it seems that no field of coincidence is formed by a unique interval, differently from the previous example.

These two examples show that there are a number of points to be better understood in the characterization of the fields of coincidence: When is a field of coincidence for a given value of α unique? Under what conditions over the supports of the distribution functions F and G is it possible to obtain a unique interval of real numbers as a field of coincidence of opinions? How does the number of intervals vary in function of α? (As to the last question, a rough upper bound for the number of intervals is $\frac{n_0(n_0+1)}{2}$, where $n_0 = n_0(\alpha) = \min\{n \in \mathbb{N} : 1 - (\frac{1}{2})^n \geqslant 1 - \alpha\}$).

Apart from the mathematical questions just mentioned, there are also some philosophical inquiries: Does it make sense to construct fields of coincidence of opinions when the supports of the distributions of X and Y are mutually exclusive? Do fields of coincidence provide a more precise interpretation of the uncertainty of a group of experts when F and G have the same support than in situations when F and G are more generic? How better is to have a group's uncompromising field instead of a group's (just mathematical) probability as a measure of its "opinion"? Does the answer to this last question depend on how radical and uncompromising are the group members? Should this be reflected on (adjoint) individual utility functions? As we can see, there is much to be studied and understood on De Finetti's fields of coincidence of opinions.

Finally, we present another interesting approach for fields of coincidence of opinions we speculate on. It corresponds to the possibility that they somehow may detect some kind of "similarity" between pairs of probability distributions. The idea of "similarity" we conceive will become clearer in the sequel, through the following example involving two situations of decision-making processes under uncertainty.

EXAMPLE 3. Consider the situations below.

SITUATION 1. Suppose you are offerred two gambles, as described below, in order to win a percentage of a large amount of money, say, \$1,000,000.00, as a prize. You should choose between the following gambles:

Gamble 1. You win a percentage x of the amount of money according to the following probability density function (p.d.f.) for x:

$$f_1 = \begin{cases} 4x & \text{if } x \in (0, 1/2) \\ 4 - 4x & \text{if } x \in (1/2, 1) \\ 0 & \text{otherwise} \end{cases}$$

Gamble 2. You win a percentage x of the money by the (p.d.f.) :

$$f_2 = \begin{cases} 1 & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

SITUATION 2. Analogously to situation 1, you are offered two gambles to choose between, namely:

Gamble 3. You win x of the money according to:

$$f_3 = \begin{cases} 1 & \text{if } x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}$$

Gamble 4. You win x of the money by:

$$f_4 = \begin{cases} 2-4x & \text{if } x \in (0, 1/2) \\ 4x-2 & \text{if } x \in (1/2, 1) \\ 0 & \text{otherwise} \end{cases}$$

In situation 1, a lot of people may prefer Gamble 1 to Gamble 2 because in the former they have a high chance of winning a good prize, around 50 per cent of the amount of money, with less risk than the latter. For the second situation, many people may prefer Gamble 4 to Gamble 3, since both options are risky and the former gives a probability of winning "almost the totality of the fortune" higher than the latter.

However, this pair of preferences, namely, Gamble 1 preferred to Gamble 2 and Gamble 4 preferred to Gamble 3, is not acceptable to the expected utility maximization paradigm (EUMP), for any non-decreasing utility function $u(\cdot)$ for money. That is, we cannot have simultaneously

$$\int_0^1 u(x) f_1(x) dx > \int_0^1 u(x) f_2(x) dx \quad \text{and}$$

$$\int_0^1 u(x) f_3(x) dx < \int_0^1 u(x) f_4(x) dx,$$

if $u(\cdot)$ is non-decreasing (this can be easily verified if we note that $f_1 - f_2 = f_3 - f_4$).

The example just presented may be seen as a continuous version of the well-known Allais' paradox, taken for granted by many EUMP opponents to refute the Bayesian standpoint. For further details on Allais' paradox, see Allais and Hagen (1979) and Savage (1954).

As Savage (1954) advocated the EUMP in the original (discrete) Allais' paradox, here we will take the side of EUMP in the aforementioned continuous version of Allais' paradox. In the sequel, we will try to detect a connection between quartets of probability distributions which may produce a continuous version of Allais' paradox (this definition is given later on), as in Example 3, and those

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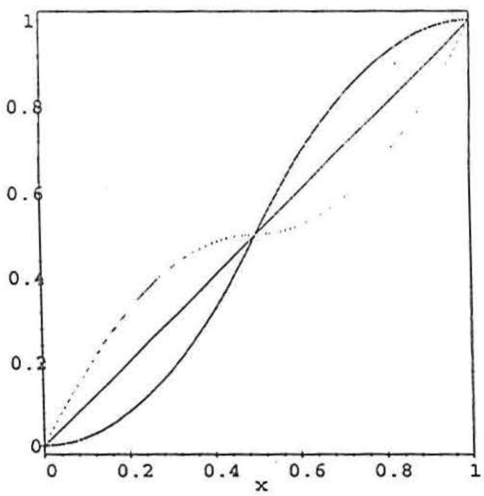

Figure 1. Caption???

whose probability distributions present, pairwise, the same fields of coincidence of opinions.

In his 1954, Savage considered simultaneously the two gambling situations in a lottery scenario involving a uniform probability distribution over the set of the numbers of the tickets, $\{1, \ldots, 100\}$, in order to defeat Allais' paradox. Now, we consider our two gambling situations simultaneously and a uniform distribution over (0,1), playing the role of the lottery scheme, as follows

For any $u \in (0, 1)$, the prize to be paid by the i-th lottery is $F_i^{-1}(u)$, $i = 1, 2, 3, 4$, where F^{-1} denotes the inverse function of F. From Figure 1, we can see that if $u \in (0, 1/2)$ then $F_4^{-1}(u) \leqslant F_3^{-1}(u) = F_2^{-1}(u) \leqslant F_1^{-1}(u)$. In this case, we have Gamble 3 preferred to Gamble 4 and Gamble 1 preferred to Gamble 2. If $u \in (1/2, 1)$ then $F_1^{-1}(u) \leqslant F_2^{-1}(u) = F_3^{-1}(u) \leqslant F_4^{-1}(u)$, so that Gamble 2 is preferred to Gamble 1 and Gamble 4 is preferred to Gamble 3. Nevertheless, in both cases we have Gamble 1 preferred to Gamble 2 if, and only if, Gamble 3 is preferred to Gamble 4. This argument throws light on this version of Allais' paradox.

It is worth mentioning that in this continuous version of Allais' paradox the set of values of u which make us indifferent between

Gamble 1(3) and Gamble 2(4), namely, {0, 1/2, 1} has probability zero, while in the discrete case the set {12, ..., 100} had probability 0.89 (see Savage, 1954).

It is interesting to note that the fields of coincidence of opinions (FCO) of f_1 and f_2, $\mathcal{C}_{12}$, is exactly the same as the FCO of f_3 and f_4, $\mathcal{C}_{34}$.This fact motivates an attempt to relate paradoxical situations (as in Example 3) and FCO. For this purpose, let us first consider the following definitions.

DEFINITION 1. *Let f_1, f_2, f_3 and f_4 be p.d.f., with $f_1 \neq f_2$ and $f_3 \neq f_4$. We say that (f_1, f_2, f_3, f_4)* produce a continuous Allais' paradox *if*

(i) $\forall\, u \in \mathcal{U} = \{u : \mathbb{R} \to \mathbb{R}^+$, *nondecreasing and continuous*$\}$ *we have*

$$\int_{\mathbb{R}} u(x) f_1(x) dx > \int_{\mathbb{R}} u(x) f_2(x) dx \Leftrightarrow \int_{\mathbb{R}} u(x) f_3(x) dx > \int_{\mathbb{R}} u(x) f_4(x) dx \quad \textit{and}$$

(ii) $\exists\, u_0, u_1 \in \mathcal{U}$ *such that*

$$\int_{\mathbb{R}} u_0(x) f_1(x) dx < \int_{\mathbb{R}} u_0(x) f_2(x) dx \quad \textit{and}$$

$$\int_{\mathbb{R}} u_1(x) f_1(x) dx > \int_{\mathbb{R}} u_1(x) f_2(x) dx.$$

DEFINITION 2. *Let f_1, f_2, f_3 and f_4 be p.d.f. with $f_1 \neq f_2$ and $f_3 \neq f_4$. We say that (f_1, f_2, f_3, f_4)* have proportional differences *if $\exists\, \alpha \in \mathbb{R}^+$ such that $f_1 - f_2 = \alpha\,(f_3 - f_4)$.*

Now we establish the following results without proof:

THEOREM 3. *If (f_1, f_2, f_3, f_4) have proportional differences and satisfy (ii) of Definition 1, then (f_1, f_2, f_3, f_4) produce a continuous Allais' paradox.*

THEOREM 4. *If (f_1, f_2, f_3, f_4) have proportional differences and $f_2 = f_3$, then the FCO for f_1 and f_2 are the same as the FCO for f_3 and f_4.*

We should note that in Theorem 4, the condition $f_2 = f_3$ is equivalent to $f_1 = f_4$.

Based on the last two results, when (f_1, f_2, f_3, f_4) satisfy (ii) of Definition 1 and $f_2 = f_3$, we speculate whether (f_1, f_2, f_3, f_4) produce a continuous Allais' paradox if, and only if, any pair of different distributions from (f_1, f_2, f_3, f_4) have the same FCO. We also suspect that if (f_1, f_2, f_3, f_4) are such that the FCO for f_1 and f_2 are the same as the FCO for f_3 and f_4, then (f_1, f_2, f_3, f_4) produce a continuous Allais' paradox. However, the reciprocal of the last assertion is not true, that is, there are quartets (f_1, f_2, f_3, f_4) that produce a continuous Allais' paradox with f_1 and f_2 having FCO different from f_3 and f_4 (this may be obtained by a little perturbation on f_2, near the points 0 and 1, in Example 3).

Our main motivation to face problems involving paradoxical situations based on FCO is to throw light on such pathologies, like violating EUMP, taking into account the "existence" of a degree of similarity between the situations offered for a gambler. We suspect that FCO may detect this kind of "confusion" in a decision process, as in Example 3. That is, a gambler who violates EUMP may be "forgiven" via recognition of similar characteristics of the options, here detected by FCO's.

As we have already emphasized, there are a lot of questions to be studied on FCO, not only in conceiving them as an "opinion" of a group of experts but also in relating them with paradoxical situations.

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