**"Pareto – Improving Default"**
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# Pareto-improving default[*]

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**Abstract**

This paper answers the question of whether non-strategic default improves welfare, not only for borrowers with uncertain future income but also for lenders with certain future endowments, relative to no default. We show that the answer is affirmative for a positive-Lebesgue-measure set of individual endowments. Numerical computations show that the size of such endowment set is larger the larger are both the risk aversion and the probability of default. Other numerical examples show that with defaultable securities lenders may finance the purchase of the latter by selling short default-free assets. This portfolio reminds those of hedge-funds such as LTCM.

# 1   Introduction

Default episodes are shown to be relevant *macroeconomic* phenomena through economic history, especially during financial crises.[1] Some of those most relevant episodes involve debtors who *cannot* repay their debt obligations as originally arranged. For example, in the last recent "subprime" crisis an important feature was the problem of many "subprime mortgage debtors"

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[1]For example, Reinhart and Rogoff (2008) show that essentially all financial crises beginning 1800 until 2003 are characterized by serial default.

not being able to pay their debt obligations, which caused the very well-known fall of major investment banks.[2]

In the aftermath of such defaults, public opinion tends to blame investment strategies that involved a (perhaps too high) fraction of the portfolio invested in defaultable securities leveraged by default-free ones.[3] In situations of generalized default, the common perception is that the mere possibility of investing in defaultable securities constitutes a threat to the stability of the financial system, especially when financial regulators allow to hedge that risk by short selling risk free bonds. This perception may become even stronger when considering households whose main purpose for lending is to smooth consumption over time without facing any risk-sharing needs. For these types of households, investing in defaultable securities may just be a bad idea.

This paper presents a theoretical framework that challenges this idea when considering the ex-ante welfare implications of *non-strategic* default (i.e., default as inability to pay) when assets are not backed by collaterals and there are no explicit penalties for defaulting. In this setting, default is just a consequence of resource uncertainty in the sense that there are objective states of nature that trigger default. When analyzed in general equilibrium, default enables the dissemination of resource uncertainty throughout the economy that originally affects only a subset of agents (namely, debtors). From this perspective, the intuition is that default, or more specifically, the existence of assets for which default is permitted, is a way of improving the overall allocative efficiency of the economy.

To capture such effect, this paper presents a very simple two-period, two-state, one-commodity, pure-endowment general equilibrium model with perfectly competitive asset markets in the tradition of the Arrow's (1953 and 1964) asset framework. There are two assets: one defaultable and another non-defaultable. As usual in this type of models, asset payoffs can vary with the two states of nature that may arise in the future period. The defaultable security is the promise of a payment in units of the commodity for one state of nature and of a zero payment for the other state, a situation which we identify to (non-strategic) *default*. The non-defaultable asset

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[2]See, e.g., the following quotation in Hellwig (2008): "*When real-estate prices began to fall* (since 2007), *deliquency rates increased dramatically. The impending* **difficulties** *in subprime mortgages and mortgage-backed securities were quickly recognized*". (Bold letters are ours).

[3]For example, in the current crisis a common reaction to the default on subprime mortgages derivatives has been to propose a ban on such derivative markets (see, for example, the newspaper-article discussion in Millman (2009) and Gow (2009)), perceiving that such defaults are not convenient to savers.

is simply a security that pays off one unit of the good in each state, i.e., it is the riskless asset.

An important assumption is related to the ability to issue securities by each consumer. One the one hand, the model assumes that the riskless asset is issued anonymously. In fact, all economic agents can short sell the riskless asset. On the other hand, the model also assumes that the defaultable security is issued by a well-defined economic agent[4]. This agent, and only this one, can therefore issue two assets, a riskless one and one on which he may default. (Note that short selling is a form of issuing a security, since short sales mean borrowing from the financial market). The state of nature that triggers default is identified with that where the issuer's endowment is low[5]. There is another consumer facing no uncertainty in his future endowment. This agent can only issue (short sell) the riskless asset. Nevertheless, this agent may find optimal, depending on endowments *and market prices*, to buy the asset with possible default in addition to trading the riskless one, becoming the lender of the agent issuing the defaultable bonds.

The analysis focuses on the comparison between equilibrium allocations for the two polar cases where default is either permitted or excluded. When only the default-free bond is allowed to be traded (and so default is excluded), equilibrium always exists, given well-known results from the incomplete markets literature, yielding the so-called *equilibrium without default.* The same literature tells that the corresponding equilibrium allocation is not Pareto efficient in general, although it is constrained-Pareto efficient (the "constraints" resulting from the exclusion of default).

From a technical point of view, introducing the defaultable security to the former economy should not necessarily imply the existence of an equilibrium where the agent with the uncertain endowment vector issues a strictly positive quantity of the asset on which he might default because of the inequality constraints that are imposed on the consumers' problems. Such an equilibrium, when it exists, is called an *equilibrium with default.* The paper shows the existence of *equilibria with default* for this simple model. The corresponding equilibrium allocations with

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[4]One may think these consumers as "company-owners" whose future productivity is uncertain, and so have risk-sharing needs. To satisfy them they issue defaultable securities to the market.

[5]Potentially, the same agent can issue several assets with the possibility of defaulting on some of them but not necessarily on all of them. For example, he can issue a riskless asset and another asset for which he will default only for some states of nature.

default are also shown to be Pareto efficient.

The idea is then to compare the ex-ante welfare at the equilibrium-with-default allocation (which is Pareto-efficient) with the equilibrium allocation in the economy without default (which is only constrained-Pareto-efficient). However, it is not obvious that for any distribution of individual endowments the former implies a Pareto-improvement relative to the latter, due to distributional effects that may be present between the two equilibria. Therefore, the paper proceeds to answer the central question, of whether the introduction of default can really make *all* agents better off. The answer to this question depends on the initial endowment vector of such agents. The main result is that there is always a subset with non-empty interior of the individual endowment space (containing the set of *equilibrium allocations without default*) such that, for all economies belonging to that subset, there exists an equilibrium allocation with default that is Pareto superior to the equilibrium allocation without default.[6]

The intuition for this result is more clear when considering an individual endowment equal to an equilibrium allocation without default. In this special case, in an economy without a defaultable security there is clearly no trade in equilibrium. At such endowment vector only the *expected* intertemporal marginal rate of substitution of the borrowers is equal to that of the lenders. However, the marginal rates of substitution between every pair of states and periods across agents differ. Thus, introducing a defaultable security improves risk sharing and consumption smoothing for all agents. This effect also works for any endowment not too far from any equilibrium allocation without default.

However, the set of endowments for which default improves ex-ante welfare exceeds those corresponding to the equilibrium allocations without default given that the main result shows that the set of such endowments has positive Lebesgue measure in the space of all possible individual endowments. The natural following question then is to know how big such a set can be. Numerical computations with constant - relative - risk - aversion preferences show that the size of such set is bigger when *both* the risk aversion coefficient *and* the probability of default are larger (but not necessarily when only one of them is larger). This numerical result suggests

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[6]The property of this component (in the set of endowments) is reminiscent of a property of the smooth Arrow-Debreu model. There, the set of equilibrium allocations—which coincides, courtesy of the two welfare theorems, with the set of Pareto optima—is contained in one component of the set of *regular smooth economies* and equilibrium is unique for endowments in that component (see Balasko, 1975a).

that financial regulations in countries with simultaneously both higher probability of low income shocks and more risk averse consumers (features that may belong to at least certain Emerging Market economies) should not consider universal bans on trading of defaultable securities as the ex-ante optimal policy. The result suggests in fact that it is important to first determine the type of default that may arise in securities markets to understand which types of default events induce ex-ante inefficiencies and which ones may improve ex-ante efficient risk allocation (even when lenders do not face risky future incomes, as it is the case in the model of this paper).

Finally, the model can generate equilibrium portfolios that resemble those of certain hedge funds (such as Long Term Capital Management, the hedge fund that went bankrupt in 1998), where these funds were long in risky securities and short in riskless ones. The paper presents numerical computations of equilibrium portfolios for logarithmic preferences. In some cases those portfolios imply indeed that the lenders purchase a positive quantity of the defaultable bond, selling short units of the riskless bond. However, the emergence of such a result depends on the numerical value of parameters of the model such as agents's beliefs on future states. This result suggests that short selling on the default-free bond (when doing this as leverage) may be part of a Pareto improving allocation for the economy, when excluding issues such as asymmetric information in the analysis. Thus, this last result suggest, at least qualitatively, that banning completely short selling (at least applied to risk free bonds, not necessarily stocks) as a form of leverage may be worse from an ex-ante welfare perspective even for lenders without needs of hedging future income risks.

As stated above, this model essentially uses a framework within the general equilibrium models focusing on financial assets, being them variations on Arrow's seminal two-period model (1953 and 1954)[7]. It is somehow surprising (but, to our knowledge, also true) that none of those well-known contributions in this literature ever reached the main result in this paper concerning the ex-ante welfare properties of default (as inability to pay) even for lenders without risk sharing motives. Thus, one way to view this paper is that it contributes to stress how "standard" general equilibrium theory with uncertainty can provide a sharp answer to such a relevant issue in financial markets.

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[7]For a sample of the huge literature devoted to these models, see Balasko and Cass (1989), Cass (1984), Geanakoplos and Mas-Collel (1989), the survey by Magill and Shafer (1991) and the textbook by Magill and Quinzii (1996).

Given the interest of default of this paper, it is also related to the more recent research agenda initiated by Shubik and Wilson (1977) and continued by Dubey and Geanakoplos (1992), Araujo et al. (2002) and Dubey et al. (2005) among others. This literature has the ambitious goal of analyzing strategic default within the incomplete competitive asset markets model. In this literature, default becomes a consequence of the optimizing behavior of economic agents, which necessitates that assets are backed by collaterals or that penalties are imposed for defaulting. However the latter approach to default does not account for the many cases where default results from the economic agent's lack of feasible alternatives. As stated above, default often occurs because resources are insufficient to cover the contractual debt. In such cases, collaterals and penalties have no effect on the decision to default. This is typically what happens in real-world financial markets with assets like bonds and, even more specifically, junk bonds.[8]

Included in the above mentioned strategic default literature, Zame (1993) deserves a special mention. That paper generates a result that seemingly sounds similar to our main result, namely, that (strategic) default improves efficiency. However, the results are different. Zame (1993) shows that, in an economy with infinite states of nature and where agents may decide not to repay their debt, but where default penalties exist for *all* future states, an increase in the latter discourages voluntary default *in any state*. This last effect, together with a sufficiently large amount of assets, makes the equilibrium allocation arbitrarily close to a Walrasian equilibrium allocation (which of course is Pareto-efficient). Clearly the last result is in contrast with the Pareto-inefficiency of an equilibrium allocation with a finite number of assets and no default. Unlike Zame's result, this paper shows that, for a set of individual endowments, introducing a security that implies default *with no penalty* in one state (but no default in the other) implies not only Pareto efficiency but also a Pareto improvement relative to the economy without such security, when there is a lender without future uncertain endowments. Such a sharp characterization of this welfare implication of default is not found in Zame (1993). Besides, the equilibrium behavior about default in both

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[8]The literature on sovereign defaults include a set of papers, starting from Grossman and Van Huyck (1988), emphasizing that even sovereign defaults some times occur just because the fiscal revenues are not enough to honor the debt as originally committed, and so the defaulter (in such a case) is essentially unpunished. (Among the papers documenting this type of default see the curious case of Phillip II in Drelichmand and Voth (2009)). However that literature says nothing about the role of this type of default on the welfare of risk-averse lenders, which is one of the twomain points of this paper.

papers are different. In Zame (1993) high enough penalties imply no default in equilibrium. In this paper default is observed in equilibrium in one of the states. Therefore, this paper shows that actual default (and not the possibility of default) improves the ex-ante welfare.

This paper is also related to the literature on financial innovation in general equilibrium. The seminal work of Hart (1975) shows that it is not always true that the introduction of a new asset in an incomplete financial markets economy improves welfare, due to the possible effects on the equilibrium relative price of goods. More recent papers show more explicit conditions under which such an addition in the asset structure implies Pareto improvement or worsening. For example, Elul (1995) shows that, on the one hand, there are certain conditions on the number of future states, assets, agents and commodities such that Hart's example of Pareto-inferior financial innovation becomes a generic property, and on the other hand, that under the same conditions there is an asset that implies a Pareto improvement. Thus, it is far from obvious to find a clear direction of the welfare properties of the introduction of a new security to an incomplete markets model.[9] Also, Cass and Citanna (1998) show a similar result when the degree of market incompleteness in the economy is sufficiently larger than the degree of heterogeneity across consumers. When there is only one commodity, Elul (1999) shows that, generically, it is always possible to find an asset that implies a Pareto improvement.

In a sense, the main result found in this paper can be viewed as providing a sharper and more precise result regarding under which conditions financial innovation really improves ex-ante welfare. In fact, this model not only shows that there is one asset that improves welfare. What the paper really stresses is that this asset implies default in one state, a default that is imposed on lenders who do not face any intrinsic motivation to share risk. Moreover, these lenders may be short selling units of the riskless security when purchasing units of the defaultable security, a characterization that is absent in the financial innovation literature referred above.

The rest of the paper is as follows. Section 2 presents the set-up including all possible securities. Section 3 characterizes the equilibrium first without the defaultable bond and then with the later, in terms of existence and Pareto efficiency. Section 4 presents the result showing the conditions implying that the addition of the defaultable bond is Pareto-improving. Section 5 presents the numerical examples. Finally section 6 presents concluding remarks as well as some

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[9]In a related work, Chen (1995) shows that frictions in financial markets (such as no short sales constraints) may play a key role in financial innovation: this innovation may improve risk sharing's investors.

suggestions for future research.

# 2   The economy

In Arrow's original model, any asset can be issued indifferently by any economic agent. We drop this assumption, which necessitates that we adapt the model accordingly. We assume that for one of the securities only one agent can issue units of that security, whereas all agents can issue units of the other security. The nominal payoff of every asset is denominated in units of a unique consumption good. Evidently, the actual payoff is equal to zero for the states of nature where the issuer is defaulting.

## 2.1   The physical environment

There are two time periods and 2 consumers, indexed by $i = 1, 2$. There is no uncertainty regarding the endowments of consumer 1. Uncertainty is limited to consumer 0's endowments. This leaves us with the formal equivalent of three "states", namely state 0 (for period 0), and states 1 and 2 depending on consumer 0's endowments in period 1. The idea is that states 1 and 2 correspond to low and high levels of resources respectively.

For simplicity, suppose that there is only one good in every state. Taking this good as the numeraire, then the spot prices of the good for states 1, 2 and 3 is just equal to 1. At some point, we will consider the standard Arrow-Debreu economy defined by the same 3 goods and the same preferences. We will then denote by $P = (P(0), P(1), P(2))$, the Walrasian price vector of that Arrow-Debreu (without assets) economy. That price $P = (P(0), P(1), P(2))$ will then be normalized by the condition $P(0) = 1$.

Preferences of consumer $i$, with $i = 1, 2$, are defined by a utility function $u_i(x_i(0), x_i(1), x_i(2))$ where $x_i = (x_i(0), x_i(1), x_i(2))$ denotes consumer $i$'s consumption of the physical goods. We assume that only strictly positive quantities of physical goods can be consumed. Let $X \equiv \mathbb{R}_{++}$ denote the strictly positive orthant of the physical commodity space $\mathbb{R}$, the consumption space of every consumer is $X^3$. We therefore have that the typical consumption bundle is $x_i = (x_i(0), x_i(1), x_i(2)) \in X^3$. Consumer $i$'s utility function is assumed to satisfy the standard assumptions of smooth consumer theory: 1) smoothness; 2) smooth monotonicity; 3) smooth

quasi-concavity; 4) closedness of indifference surfaces in the contingent and dated commodity space $\mathbb{R}$. (See, e.g., Balasko, 1988) Also, we assume that the utility functions are time and state separable à la Savage. In other words, consumer $i$'s utility can be written as

$$u_i\left(x_i(0), x_i(1), x_i(2)\right) = u_i\left(x_1(0), 0\right) + \alpha u_i\left(x_i(1), 1\right) + (1 - \alpha)u_i\left(x_i(2), 1\right) \tag{1}$$

where $\alpha \in [0, 1]$ is the probability of state $s = 1$.

On the other hand, consumer $i$ is endowed with the commodity bundle $\omega_i = (\omega_i(0), \omega_i(1), \omega_i(2)) \in X^3$, with $\omega_i(1) = \omega_i(2)$ for $i \geq 1$. (Recall that there is no uncertainty regarding the resources of consumer $i$, with $i \geq 1$, in period 1.) We essentially assume that, for consumer 0, $\omega_0(1) < \omega_0(2)$, while for agent 1, the assumption is that $\omega_1\left(1\right) = \omega_1\left(2\right) \equiv \omega_1^1$ and that

$$\omega_1\left(0\right) > \omega_1^1 \tag{2}$$

In other words, agent 0's endowment in period 1 is uncertain, receiving more goods in state 2 than in state 1, while agent 1 faces no uncertainty in his future (period 1) endowment but the latter is lower than lender's period-0 endowment. This assumption will imply that in equilibrium the lender has an intrinsic need for consumption smoothing, in the sense of having incentives to *save* in period 0. Let $r \equiv \left(\omega\left(0\right), \omega\left(1\right), \omega\left(2\right)\right)$ denote the vector of aggregate endowments for this economy.

## 2.2   Assets

### 2.2.1   The riskless asset

Every consumer can issue a riskless asset. An obvious no arbitrage condition implies that the prices of the riskless assets issued by different economic agents must be proportional. This condition is equivalent to having a unique riskless asset that is issued anonymously. The payoffs of a unit of the riskless asset consist of one unit of numeraire in states 1 and 2. We denote by $q_1$ the price (in period 0) of this asset and by $b_i^1$ its "consumption" by consumer $i$, with $i \geq 0$. In the case where the possibility of default is excluded in the economy, then the riskless asset is the **unique asset.**

### 2.2.2 The asset with default

If default is permitted, this takes the form of consumer 0 being allowed to issue the asset characterized by payoffs equal to 0 for state $s = 1$ and to one unit of numeraire for state $s = 2$. We denote by $q_0$ the price (in period 0) of that risky asset and by $b_i^0$ the "consumption" of the risky asset by consumer $i$, with $i \geq 0$. Since this asset is necessarily issued by consumer 0, we must have $b_0^0 \leq 0$ and $b_i^0 \geq 0$ for $i \geq 1$.

# 3    Benchmark: the economy without default.

This section considers the benchmark case where only the riskless asset is traded. The following section analyzes the economy with both assets, the riskless asset and the asset with default. Recall that the riskless asset is issued anonymously and by possibly both agents 0 and 1.

## 3.1    No-default equilibrium: main elements

First we present the set up of the optimal consumer's problem. For any agent $i \geq 0$, consumer $i$ faces one budget constraint for every state of nature $s = 0, 1, 2$. In the model with default, there are two assets and the budget constraints take the form

$$x_i(0) - \omega_i(0) = -q_1 b_i^1 \tag{3}$$
$$x_i(1) - \omega_i(1) = b_i^1$$
$$x_i(2) - \omega_i(2) = b_i^1$$

Clearly, this consumer's maximization problem has always a solution for any $i \geq 0$ and any price system $(p, q_1)$.

Second, in an equilibrium for the economy defined by the endowment vector $\omega = (\omega_0, \omega_1)$ total supply of the physical goods and assets must equal total demand of them. This gives us the following equilibrium condition:

$$\sum_i x_i(s) = \sum_i \omega_i(s), \ s \in \{0, 1, 2\} \tag{4}$$
$$\sum_{i \geq 0} b_i^1 = 0.$$

## 3.2 No-default equilibrium: analysis

Note that this model is a standard general equilibrium model with one asset with real payoffs. The rank of the payoff matrix is equal to one. It then follows from, e.g., Duffie and Shafer (1985), that equilibrium always exists. In addition, generically on endowments, equilibrium is locally isolated but there may exist multiple equilibria.

Before starting the analysis, it is convenient to introduce a formal tool for the analysis. Let $V_i$ be the plane in the commodity space $\mathbb{R}$ consisting of the points $\omega_i + b_i^1(-q_1, 1, 1)$ where $q_1$ and $b_i^1$ are varied in the set of real numbers. It follows from the budget constraints satisfied by consumer $i$ that the consumption bundle $x_i = (x_i(0), x_i(1), x_i(2))$ in the model without default must belong to the plane $V_i$. Then, all consumption allocations included in $V_1$ must satisfy the equality $x_1(1) = x_1(2)$, given the assumption on agent 1's endowments. The equilibrium allocation $x = (x_0, x_1)$ is therefore *constrained Pareto efficient*, the constraint being that consumer $i$'s allocation $x_i$ belongs to the plane $V_i$. In general, the equilibrium allocations for the model without default are not Pareto efficient.

### 3.2.1 A geometric tool: three-dimensional extension of the Edgeworth box

The commodity space is the ordinary three dimensional space $\mathbb{R}^3$ of Solid (i.e., three-dimensional Eucildean) geometry. This enables us to use the three dimensional analog of the Edgeworth box. The minor difficulty due to having an additional dimension is compensated by the intuition brought by the geometric formulation.

As in the standard Edgeworth box, the vector of total endowments $r \in \mathbb{R}^3_{++}$ is fixed. We have two coordinate systems. The first coordinate system is centered at the point $O_0$ and is used to represent consumer 0's resources, consumption, and preferences. The second coordinate system is centered at $O_1$, the extremity of the vector of total resources $r \in \mathbb{R}^3$ in consumer 0's coordinate system. The coordinate axes for consumer 1 are parallel and oriented in the opposite direction to those of consumer 0. Let $x = (x_0, x_1)$ be a feasible allocation, i.e., an allocation such that $x_0 + x_1 = r$. Let $M$ be the point in $\mathbb{R}^3$ whose coordinates in consumer 0's coordinate system are $x_0$. Then, the coordinates of point $M$ in the coordinate system of consumer 1 are then equal to $x_1$.
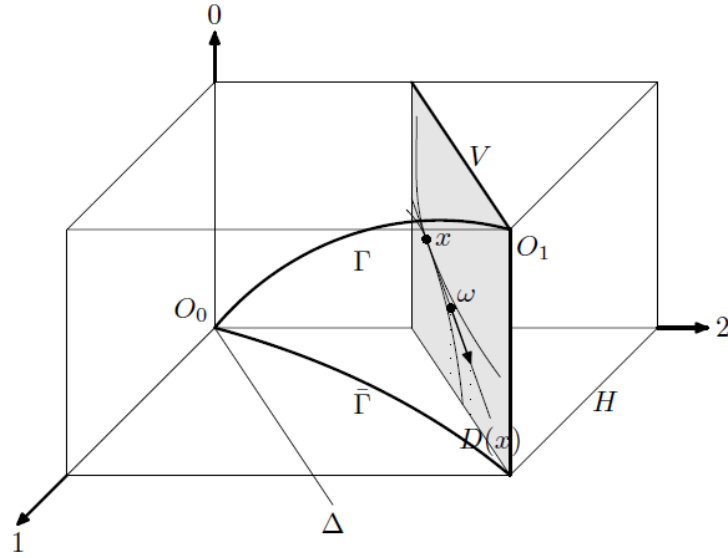
Figure 1: The no-default case: 3-dimensional view

### 3.2.2 Pareto optima and the contract curve in three dimensions

It will become very useful to graphically characterize the Pareto-efficient consumption alloca-
tions. To do this, note that consumer 0's preferences are represented by a collection of in-
difference *surfaces* in consumer 0's coordinate system. Similarly, consumer 1's preferences are
represented by a collection of indifference surfaces in consumer 1's coordinate system. An allo-
cation $y = (y_0, y_1)$ then is a Pareto optimum if the point $M$ that represents $y = (y_0, y_1)$ (i.e.,
the coordinates of $M$ in consumer 0's coordinate system are equal to $y_0$) is the contact point
of the indifference surfaces of the two consumers through that point. The budget plane $F(y)$
associated with the Pareto optimum $y = (y_0, y_1)$ is the common tangent plane to the two indif-
ference surfaces that pass through that point. Also, the **supporting Walrasian price vector**
$P = (P(0), P(1), P(2))$ for the Pareto optimum $y = (y_0, y_1)$ is perpendicular to the budget
plane $F(y)$. The allocation $y = (y_0, y_1)$ is then the **Walrasian equilibrium allocation** asso-
ciated with the **Walrasian price vector** $P = (P(0), P(1), P(2))$ and any endowment vector
$\omega = (\omega_0, \omega_1)$ in that budget plane $F(y)$.

### 3.2.3 The vertical plane of endowments and of equilibrium allocations without default

Let the horizontal plane $H$ be defined by the coordinate axes. The plane $H$ represents the goods consumed in states 1 and 2 by consumer 0. We also consider a line $\Delta$ that is the diagonal of these two axes. Along this diagonal, $x_0(1) = x_0(2)$. We have defined in subsection 3.2 the planes $V_0$ and $V_1$. In the three-dimensional Edgeworth box these two planes become essentially one vertical plane, a plane that contains the point $O_1$ and that is parallel to the diagonal line $\Delta$. We denote that plane by $V$. In addition, it follows from the condition $\omega_1(1) = \omega_1(2)$ that reflects the certainty of consumer 1's resources that the endowment vector $\omega = (\omega_0, \omega_1)$ can be any point in the vertical plane $V$. (The endowment vector $\omega = (\omega_0, \omega_1)$ is actually contained in the box of $V$ defined by the two consumers' coordinate axes if all endowments are to be $> 0$).

In addition to the endowment vector $\omega = (\omega_0, \omega_1)$ belonging to the vertical plane $V$, the equilibrium allocations without default also belong to the plane $V$ because, since consumer 1 has no uncertainty on period 1 endowments and can only trade in the riskless security, the obvious result is that this same consumer faces no uncertainty in his period 1 consumption. Therefore, the study of the model without default is reduced to the study of the model defined by the restriction of the preferences of consumers 0 and 1 to the plane $V$. Since endowments and allocations also belong to that plane $V$, they define a standard two-dimensional Edgeworth box.

The allocation $x = (x_0, x_1)$ is an *equilibrium allocation without default* if and only if $x$ is an equilibrium allocation associated with $\omega = (\omega_0, \omega_1) \in V$ for the two-dimensional Edgeworth box defined by the vertical plane $V$ and the restriction of the two consumers' preferences to that plane. Existence of an *equilibrium without default* then follows readily in the current case from the existence of equilibrium for a standard Arrow-Debreu model with two goods and two consumers[10].

It is obvious from figure 1 that the *equilibrium allocation without default* $x = (x_0, x_1)$ is Pareto optimal with respect to all allocations that belong to the vertical plane $V$. It is therefore a *constrained Pareto optimum.* In general, however, this allocation is not a Pareto optimum

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[10]Incidentally, this technique of proof could easily be extended to work for the general case of any number of consumers.
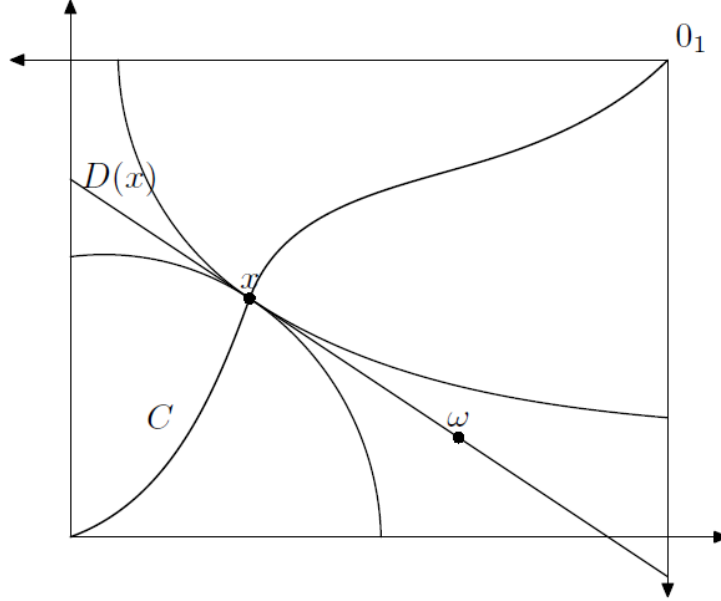
Figure 2: The no-default case: 2-dimensional view

in the three dimensional space because the two consumers' indifference surfaces through the equilibrium allocation without default $x = (x_0, x_1)$ do not have the same tangent plane at this point even if the "budget line" defined by the asset without default links the allocation $x$ and the endowment $\omega$ and is tangent to both indifference surfaces at $x = (x_0, x_1)$.

We denote by $C$ the "contract curve" for the two-dimensional Edgeworth box defined by the vertical plane $V$ and the restrictions of the two consumers' preferences to that plane. The "contract curve" consists of *constrained Pareto optima:* they belong to the plane $V$ and cannot be Pareto dominated by other points of $V$. Incidentally, varying the plane $V$ (which amounts to varying the distribution of total resources between consumer 0 and 1) generates the surface of *constrained Pareto optima* in the three-dimensional Edgeworth box.

# 4   The economy with default

This section reintroduces the defaultable security in the analysis. As stated below, several differences arise in equilibrium. The graphical tool introduced in the last section will become also a very powerful tool to characterize equilibria when both securities are traded (and where only agent 0 can issue the defaultable asset).

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## 4.1   Default equilibrium: main elements.

As above, we introduce the set up for the optimal decision for consumer $i = 1, 2$. In the model with default, there are two assets and the budget constraints take the form

$$x_i(0) - \omega_i(0) = -q_0 b_i^0 - q_1 b_i^1 \tag{5}$$

$$x_i(1) - \omega_i(1) = b_i^1$$

$$x_i(2) - \omega_i(2) = (b_i^0 + b_i^1)$$

where, necessarily, $b_0^0 \leq 0$ and $b_i^0 \geq 0$ for $i \geq 1$. Consumer $i$'s demand $(x_i, b_i)$ in this case then maximizes the utility $u_i(x_i(0), x_i(1), x_i(2))$ subject to the budget constraint (5). The consumer's maximization problem in the model with default has always a solution for any $i \geq 0$ and any price system $(p, q_0, q_1)$.

Therefore, the price system $(p, q_0, q_1)$ is an equilibrium with default of the economy defined by the endowment vector $\omega = (\omega_0, \omega_1)$ if there is equality between total supply and demand of the physical goods and assets, and *if the proper sign constraints are satisfied by the consumers' portfolios.* This gives us the following equilibrium condition:

$$\sum_i x_i(s) = \sum_i \omega_i(s), \ s \in \{0, 1, 2\} \tag{6}$$

$$\sum_{i \geq 0} b_i^0 = 0, \qquad b_i^0 \geq 0 \text{ for } i \geq 1,$$

$$\sum_{i \geq 0} b_i^1 = 0.$$

## 4.2   Default equilibrium: analysis

The model with default is simply the model with the two assets combined with the restriction that the portfolios must satisfy at equilibrium the sign constraints $b_i^0 \geq 0$ for all $i \geq 1$. The equilibrium condition $\sum_{i \geq 0} b_i^0 = 0$ then implies the inequality $b_0^0 \leq 0$. We exclude the trivial case where $b_i^0 = 0$ for all $i$ since no asset with default is then issued at equilibrium. Therefore, from now on, we impose the condition $b_i^0 > 0$ for some $i \geq 1$. It is when these inequalities are satisfied that we have an *equilibrium with default.*

It can very well be that no *equilibrium with default* exists for a given endowment vector $\omega = (\omega_i)_{i \geq 0}$. In such a case, the economy settles to an equilibrium without default. The issue,

therefore, is whether *equilibrium with default* can exist and to get a better understanding of when such an equilibrium may exist. We address this question only for the simplest version of our model. Like the case without default, this analysis benefits from the geometrical tool introduced above.

## 4.3    Equilibria with default

We now prove the existence of *equilibria with default* for the case of the leading example.

**Theorem 1** *Under assumptions A and B in the leading example, all Walrasian equilibrium allocations associated with any endowment vector $\omega = (\omega_0, \omega_1) \in V$ that is not a Pareto optimum satisfy the portfolio sign conditions that identify them to equilibria with default. The latter are Pareto efficient.*

**Proof**. Before starting with the proper proof, consider the case where the sign constraints $b_i^0 \geq 0$ for $i \geq 1$ that must be satisfied by the individual portfolios at equilibrium (the equilibrium condition $\sum_{i \geq 0} b_i^0 = 0$ then implies $b_0^0 \leq 0$) are ignored. In this case, we have a standard general equilibrium model with two independent assets because the rank of the payoff matrix is equal to two. This model has therefore an equilibrium for any endowment vector $\omega = (\omega_i)_{i \geq 0}$.

Note that if we ignore the sign of the portfolios—in other words, we forget provisionally the assumption that the risky asset 0 can be issued only by consumer 0,—the elimination of $b_i^0$ and $b_i^1$ between the three budget constraints yields the unique budget constraint that involves only the physical goods:

$$(x_i(0) - \omega_i(0)) + (q_1 - q_0)(x_i(1) - \omega_i(1)) + q_0(x_i(2) - \omega_i(2)) = 0.$$

Therefore, the price vector $(q_0, q_1)$ is an *equilibrium price vector with default* (excluding the consideration of the sign constraints on portfolios) if and only if the Walrasian price vector (cf. Section 2.1) $P = (P(0), P(1), P(2))$ where $P(0) = 1$, $P(1) = (q_1 - q_0)$ and $P(2) = q_0$ is an equilibrium price vector of the Arrow-Debreu model defined by the endowment vector $\omega = (\omega_0, \omega_1)$.

Given $\omega = (\omega_0, \omega_1)$ in the vertical plane $V$, a Walrasian equilibrium exists by the standard existence theorems. The corresponding Walrasian equilibrium allocation $y = (y_0, y_1)$ is a Pareto

optimum. At this stage, the question that remains open is whether the portfolio sign constraints that define an *equilibrium with default* are satisfied at such Walrasian equilibrium. The remaining part of the proof demonstrates that the answer to the former question is affirmative.

It is clear that if $\omega = (\omega_0, \omega_1) \in V$ is already a Pareto optimum (in the sense of the three-dimensional Edgeworth box), then the corresponding equilibrium allocation without default $x$ is uniquely defined and equal to $\omega$, as is the equilibrium allocation $y$ in the three-dimensional Edgeworth box. There is no non trivial equilibrium allocation with default.

Let $\omega = (\omega_0, \omega_1) \in V$ that is not a Pareto optimum. Let $y = (y_0, y_1)$ be a Walrasian equilibrium allocation associated with $\omega = (\omega_0, \omega_1) \in V$. This point $y = (y_0, y_1)$ is therefore different from $\omega = (\omega_0, \omega_1)$. Let $F(y)$ be the plane through $y$ that is tangent to the two indifference surfaces that pass through $y$.

Let $P = (P(0), P(1), P(2))$ be the Walrasian price vector perpendicular to the plane $F(y)$ with the normalization $P(0) = 1$. We have $P(0) = 1$, $P(1) = q_1 - q_0$ and $P(2) = q_0$, from which follows $q_0 = P(1)$ and $q_1 = P(1) + P(2)$. Let $\vec{a}_0$ and $\vec{a}_1$ be the vectors

$$
\vec{a}_0 = \begin{bmatrix} -P(1) \\ 0 \\ 1 \end{bmatrix} \qquad \vec{a}_1 = \begin{bmatrix} -P(1) - P(2) \\ 1 \\ 1 \end{bmatrix}
$$

These vectors are just the representation of the assets with and without default respectively. The tangent plane $F(y)$ is the plane parallel to the vectors $\vec{a}_0$ and $\vec{a}_1$ through the point $y = (y_0, y_1)$. The endowment vectors $\omega = (\omega_0, \omega_1)$ admits the allocation $y = (y_0, y_1)$ as an *equilibrium allocation with default* if it belongs to the half-plane $F_+(y)$ of $F(y)$ that is generated by the vector $b_0^0 \vec{a}_0 + b_0^1 \vec{a}_1$ with $b_0^0 > 0$ while $b_0^1$ can have any sign.

Let $\bar{y} = (\bar{y}_0, \bar{y}_1)$ denote the orthogonal projection of $y = (y_0, y_1)$ into the horizontal plane $H$ and let $\Delta(\bar{y})$ denote the line parallel to the diagonal $\Delta$ through $\bar{y}$. It follows from the assumptions above that $\omega_0(1) < \omega_0(2)$. The projection of the half-plane $F_+(y)$ into the horizontal plane $H$ is therefore the half-plane that contains the point $O_1$ and that is bounded by the line $\Delta(\bar{y})$.

It also follows from assumptions above that the projection $\bar{y}$ of the Pareto optimum $y = (y_0, y_1)$ is a Pareto optimum for the fictitious economy made of the goods delivered in states 1 and 2. Let $\bar{\Gamma}$ denote the set of Pareto optima for this two-good economy. Again, but now in the horizontal plane $H$, we have an Edgeworth box. Let $\Delta'$ denote the projection of the vertical

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plane $V$ into $H$. The line $\Delta'$ is the parallel to $\Delta$ that passes through the point $\overline{O}_1$, the projection of $O_1$ in the horizontal plane $H$.

The tangents to the indifference curves of consumer 0 at points of $\Delta$ are perpendicular to the vector $(\alpha, 1 - \alpha)$. Similarly, the tangents to the indifference curves of consumer 1 at points of $\Delta'$ are also perpendicular to the vector $(\alpha, 1 - \alpha)$. Combined with the convexity of the two consumers' indifference curves, this implies that the tangency points of the indifference curves belong necessarily to the strip of the plane $H$ determined by the lines $\Delta$ and $\Delta'$. This proves that the contract curve $\bar{\Gamma}$, the projection in $H$ of the 3D contract curve $\Gamma$ is contained in the half-plane delimited by the line $\Delta'$ and containing the point $O_0$. This proves that all points $\omega = (\omega_0, \omega_1) \in V$ belong to the half-plane $F_+(y)$, for any $y \in \Gamma$. This ends the proof that $y$ is therefore an *equilibrium with default* for $\omega \in V$. $\blacksquare$

This existence result essentially implies that the portfolio constraints defined by the restriction that only agent 0 can issue the defaultable bond (and so agent 1 can only purchase this asset) are not binding at equilibrium. The main reason is the endowment pattern assumed for both agents: agent 0 does want to issue a defaultable asset given his risky endowment in period 1, while agent 1 wants to save in period 0 by purchasing the defaultable asset.

# 5   When default is Pareto superior to no default

## 5.1   The main result

The last section shows that, for every endowment vector $\omega = (\omega_0, \omega_1)$ included in the set of all possible endowment vectors $V$ that is not a Pareto optimum, there exists an equilibrium without default and also an equilibrium with default. The equilibrium allocation with default is clearly Pareto efficient while the equilibrium allocation without default is only constrained-Pareto efficient, but not Pareto efficient. Nevertheless, this welfare difference does not imply that, for given $\omega$, every consumer is going to be ex-ante better off with the equilibrium with default because distributional issues may make some consumers less well off at the Pareto optimum. Note that even ignoring the portfolio sign constraints it is not true in general that the equilibria of such a model are Pareto superior to those of the first model. In fact, the study of how the equilibria of these two models are related is a special case of the problem of financial innovation.

It follows from that literature that those relationships can be quite complex (see the discussion in the introduction).

The next proposition is the main result of the paper, showing that there is always a set of endowments (with non-empty interior) in $V$ such that all agents's ex-ante welfare (for *both* agent 0 and agent 1) increases when going from a default-free equilibrium to a default equilibrium for each endowment in that set:

**Proposition 2** *Let $D(x)$ be the common tangent at $x$ in $C$ to the indifference curves of the two consumers (whose preferences are restricted to $V$) through the point $x$. The subset of the line $D(x)$ that consists of the endowment points such that the equilibrium allocation without default is Pareto dominated by an equilibrium allocation with default contains a segment with non-empty interior.*

**Proof**. Recall that we denote by $C$ the curve that consists of the constrained Pareto optima in the vertical plane $V$. The curve $C$ is also the set of *equilibrium allocation without default* associated with endowments in the vertical plane $V$. Note that the line $D(x)$ is the common tangent to the two consumers' indifference surfaces through the point $x$ in the three dimensional space. The line $D(x)$ consists of the endowment vectors $\omega = (\omega_0, \omega_1)$ for which $x = (x_0, x_1)$ is an *equilibrium allocation without default*. We can now characterize the points $\omega = (\omega_0, \omega_1)$ of the line $D(x)$ that, as endowments, have *equilibrium allocations with default* that are Pareto superior to the allocation $x = (x_0, x_1)$.

Let $\Gamma$ be the contract curve consisting of the Pareto optima in the three-dimensional space. Let $P_0$ (resp. $P_1$) be the point of $\Gamma$ with utility for consumer 0 equal to $u_0(x_0)$ (resp. for consumer 1 equal to $u_1(x_1)$). The points $P_0$ and $P_1$ define an arc of the contract curve $\Gamma$. This arc consists of the allocations that are three-dimensional Pareto optima and that are Pareto superior to the allocation $x = (x_0, x_1)$.

Let $y = (y_0, y_1)$ be some arbitrary point of the arc $P_0 P_1$ of the curve $\Gamma$. The point $y = (y_0, y_1)$ represents an equilibrium allocation (in the standard Walrasian sense) associated with the endowment point $\omega = (\omega_0, \omega_1) \in D(x)$ if and only if $\omega = (\omega_0, \omega_1)$ belongs to the budget plane $F(y)$, associated with the Pareto optimum $y = (y_0, y_1)$.
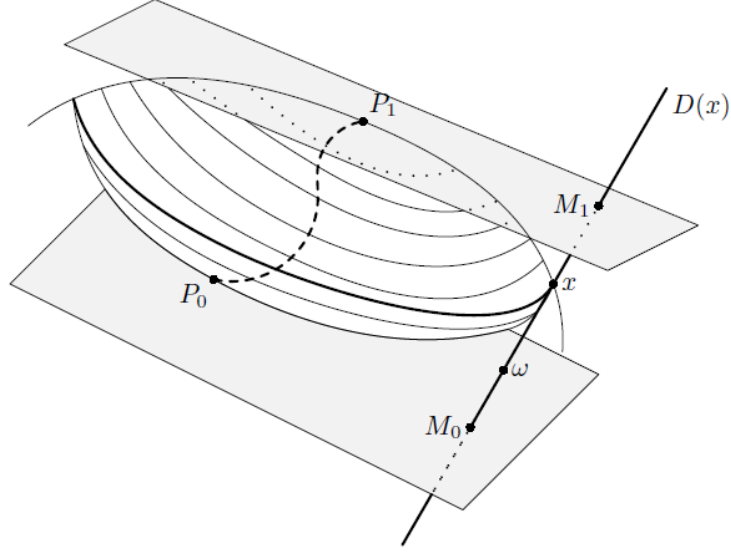
Figure 3: Pareto improvement associated with risky assets.

Let us consider the intersection of the plane $F(y)$ with the line $D(x)$. There are two possibilities: 1) For some $y$, the plane $F(y)$ contains the line $D(x)$; 2) For all $y$ belonging to the arc $P_0 P_1$, the plane $F(y)$ either intersects the line $D(x)$ at just a point or is parallel to $D(x)$.

In the first case, for any endowment vector $\omega = (\omega_0, \omega_1)$ that belongs to $D(x)$, the equilibrium allocation without default $x = (x_0, x_1)$ is Pareto dominated by $y = (y_0, y_1)$. The allocation $y = (y_0, y_1)$ is an *equilibrium allocation with default* for **all the points** of the line $D(x)$.

In the second case, let us compactify the line $D(x)$ by adding a point at "infinity." The map $y \to \{D(x) \cup \{\infty\}\} \cap F(y)$ is then defined on the arc $P_0 P_1$ and is continuous. Its image of the connected arc $P_0 P_1$ is a connected subset of $\{D(x) \cup \{\infty\}\}$, a set that is homeomorphic to the circle $S^1$. The connected subsets of $S^1$ are the intervals. Therefore, the image of this map is an interval of the line compactified by a point at infinity $\{D(x) \cup \{\infty\}\}$. This interval contains the points $M_0$ and $M_1$ that are the intersection of the budget planes $F(P_0)$ and $F(P_1)$ with the line $\{D(x) \cup \{\infty\}\}$ and also contains the point $x = (x_0, x_1)$. In addition, it follows from the convexity and orientation of the indifference surfaces of the two consumers through the point $x = (x_0, x_1)$ that $x$ belongs to the segment $M_0 M_1$.

It then follows from the connectedness property of the image that all the endowments that belong to the segment $M_0 M_1$ are such that there exists an equilibrium allocation with default that is Pareto superior to the equilibrium allocation without default $x = (x_0, x_1)$. ∎
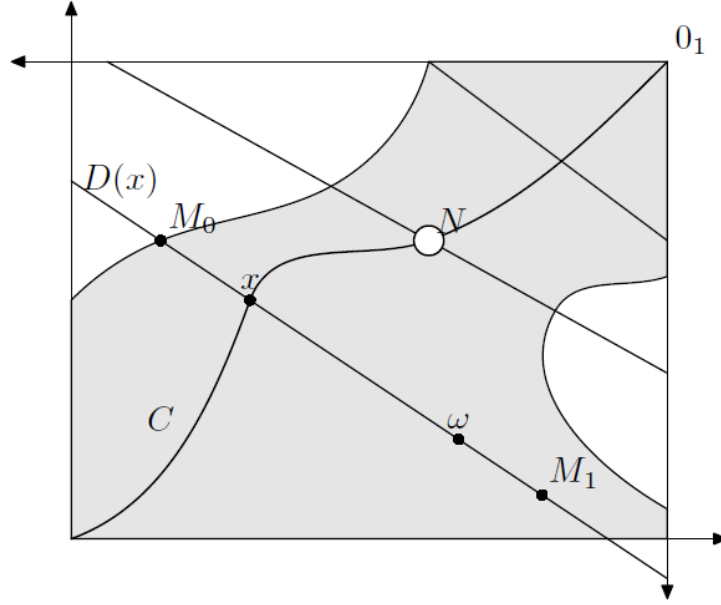
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Figure 4: Economies where an equilibrium with default dominates the equilibrium without default

The economic meaning of this result is clear. Endowments in this segment correspond to those "close to" no default equilibrium allocations. This means that, if the endowments for both agents 0 and 1 are such that when only the riskless asset is traded, the volume of trading is sufficiently low (close to 0), then this means that opening a new market where the defaultable security is traded improves ex-ante welfare, not only to agent 0 (the "borrower") who has intrinsic risk sharing needs, but also to agent 1, who only wants to smooth consumption over time, except in the degenerate case where $x = (x_0, x_1)$ is already a Pareto optimum instead of just being a constrained Pareto optimum. Endowments corresponding to exact no-default-equilibrium allocations satisfies the equality between the intertemporal marginal rate of substitution (between consumption of periods 0 and 1) for agent 1 and the *expected* intertemporal marginal rate of substitution for agent 0. When endowments satisfy conditions close to this one then one expects that this default without stigma (in state 1) implies a Pareto improvement.

In fact, the proposition above shows that those endowments may be "not close enough" to those corresponding to no default equilibrium allocations. The graphical argument shows that the size of the segment of $D(x)$ containing such endowments may depend on the curvature of the indifference surfaces of agents 0 and 1. The size of the set of endowments where Pareto

improvement holds can be seen in Figure 4. Note that the curve $C$ contains at least one three-dimensional Pareto optimum obtained by intersecting the three-dimensional contract curve $\Gamma$ with the vertical plane $V$. Let $N$ be such a Pareto optimum. When the point $x$ of $C$ tends to $N$, then the line $D(x)$ tends to a limit position which is the intersection of the budget plane that supports the Pareto optimum $N$ and the vertical plane $V$. In addition, the intersection points $M_0$ and $M_1$ tend also to limit positions. The latter are generally different from the point $N$.

By varying the constrained Pareto optimum $x = (x_0, x_1)$ along the contract curve $C$, the interval defined in Proposition 2 generates the set of endowment points in the vertical plane $V$ for which there exists an equilibrium allocation with default that Pareto dominates the equilibrium allocation without default. Provided that we include in that definition the case where the equilibrium allocation without default is already Pareto efficient (in which case it is trivially improved by the introduction of default), then we see that the set of economies where the introduction of default is a Pareto improvement over no default contains the curve $C$. In addition, this set has a non empty interior and contains the subset bounded by the curves generated by the points $M_0$ and $M_1$.

## 5.2   How "likely" is that default leads to Pareto improvement? Numerical exercises.

Given the general result in proposition (2), a legitimate question that arises is, for given aggregate endowments and preferences for both agents, how *big* is the size of individual endowments for which both borrowers and lenders are better off with default than without it. Given that this Pareto improvement result is not necessarily universally valid (although the proposition shows that the Lebesgue measure of the above mentioned set is strictly positive) then it is clear that, the bigger is the size of such set, the more general is this feature.

Both figure 3 and the proof of proposition 2 indicate the steps to characterize such a set. For every possible value of the expected utility of one of the consumers (say, agent 0), compute the allocation corresponding to the constrained-Pareto efficient allocation consistent to that utility (point $x$ in figure 3), and also compute the expected utility reached by the other agent (agent 1) at that allocation. Use the two values of the expected utility to obtain two Pareto-efficient allocations, each one corresponding to each utility value (points $P_0$ and $P_1$ in figure 3). After

computing the gradient in each of the two allocations, use the gradients two project the two allocations onto the plane that contains all possible individual endowments (i.e., endowments such that $\omega_1(1) = \omega_1(2)$) to get the allocations such as $M_1$ and $M_0$. By changing the values of the utility originally chosen (from the minimum value to the maximum possible value) the set can be generated in this way.

Unfortunately, a general characterization of such a set using the above-mentioned procedure is not possible. Thus, two answer the question on the size of the set of Pareto-improving endowments makes necessary at least the parametrization of the utility function to get a numerical characterization of the equilibria. The chosen parametrization corresponds to the CRRA case (constant relative risk aversion), i.e.:

$$ u_i(x_i) = \begin{cases} \frac{x_i^{1-\sigma}-1}{1-\sigma} \text{ if } \sigma \neq 1,\ \sigma > 0 \\ \ln x_i \text{ if } \sigma = 1 \end{cases} $$

Just for illustration, the values for the aggregate endowments are arbitrarily chosen in the following values: $\omega(0) = 4$, $\omega(1) = 1$, $\omega(2) = 3$.

We numerically compute the boundaries for the set of individual endowments for which default implies a Pareto improvement in the context of the simple economy considered in this paper. The following figure shows this set when $\sigma$ is equal to 0.5 and $\alpha$ (the probability of state

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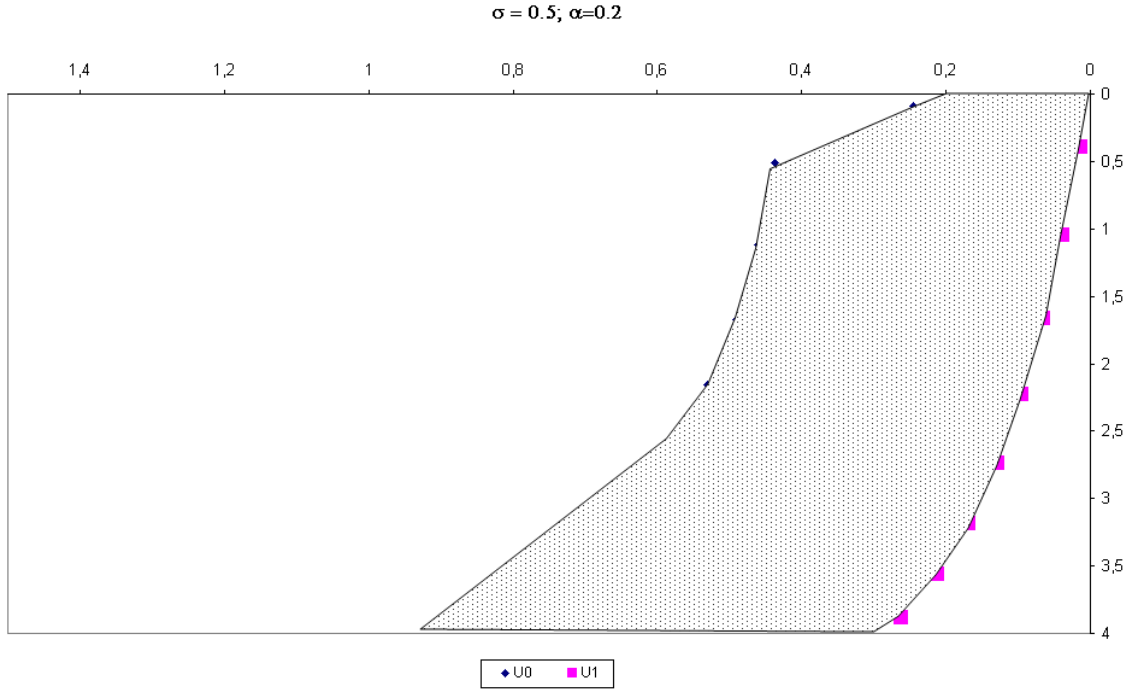1, in which default would occur if a defaultable security is traded) is equal to 0.2.



Figure 5: set of individual endowments such that default Pareto improves ex-ante welfare when $\alpha = 0.2$ and $\sigma = 0.5$.

The whole rectangle in figure 5 represents the set $V_1$ in this numerical example. The set of endowments in $V_1$ such that default implies Pareto improvement is included in the shaded-dotted area in $V_1$. The message of this figure is that, at least for the proposed parameter values, the size (Lebesgue measure) of the set of individual endowments for which Pareto improvement occurs is far from being negligible. However, the second natural question is whether the endowments on the boundary of such a set depend on parameters such as risk aversion or the probability of default. In fact, the characterization of the set coming from the proposition 2 and figure 3 suggests that the size of the set of default-Pareto-improving endowments depend on the curvature of the indifference surfaces, and the structure of the model suggests that such a curvature is related to the risk aversion or the probability of default.

It turns out that, for the value of the probability of state 1 equal to 0.2, higher values of $\sigma$ (the relative-risk-aversion coefficient) does not necessarily increase the size of such a set.

Computations[11] done for values of $\sigma$ equal to 1.1 and 2 show that the set of such endowments (on the plane $V_1$) *shifts* downwards to the right relative to the benchmark case shown in figure 5.2. Also, computations obtained for $\sigma = 0.25$ imply a set shifted upwards and to the left relative to the the set in figure 5.2.

We also computed such a set for the case of $\alpha = 0.4$ instead of 0.2 (maintaining the value of $\sigma$ in 0.5). This exercise essentially tries to link the set of endowments for which default implies a Pareto improvement with the probability that such a default occurs (when there is a defaultable security). However, such increase in the probability of default does not increase the size of the set either. Instead, it shifts it to the left and upwards.

However, when considering *simultaneously both* a higher value of the risk aversion coefficient *and* a higher value of the probability of default the size of the set undoubtedly increases. The following figure shows the set for the same values of aggregate endowments and when the coefficient of relative risk aversion is 0.75 and the probability of state 1 is 0.4.
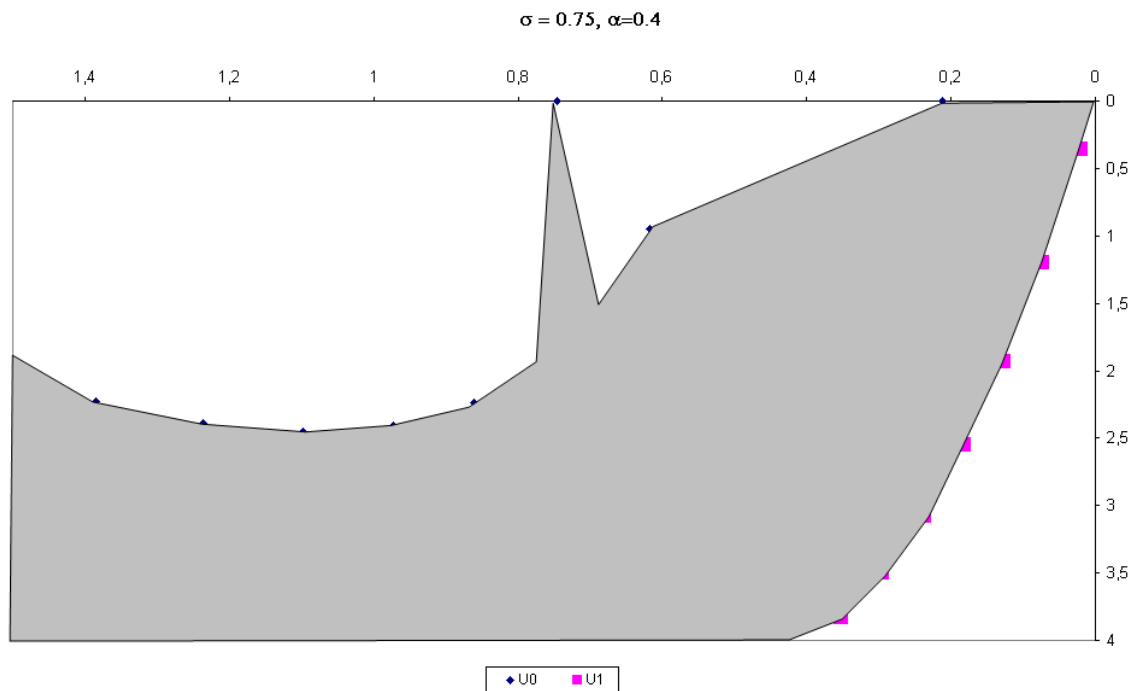


Figure 6: set of individual endowments such that default Pareto improves ex-ante welfare when $\alpha = 0.4$ and $\sigma = 0.75$.

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[11]For brevity the figures are not shown here. Such figures are available upon request.

In figure 6 the set of endowments for which default improves ex-ante efficiency is given by the dark-grey shaded area. Provided that the size of the plane $V_1$ is the same in both figures (5.2) and (5.2), just by inspection of the boundaries of both sets it is obvious that the green surface in figure 5.2 is larger than the light-blue one in figure 5.2.

This feature is also seen in the following figure, which considers the same value of $\alpha$ (0.4) and a value of risk aversion equal to 0.9.
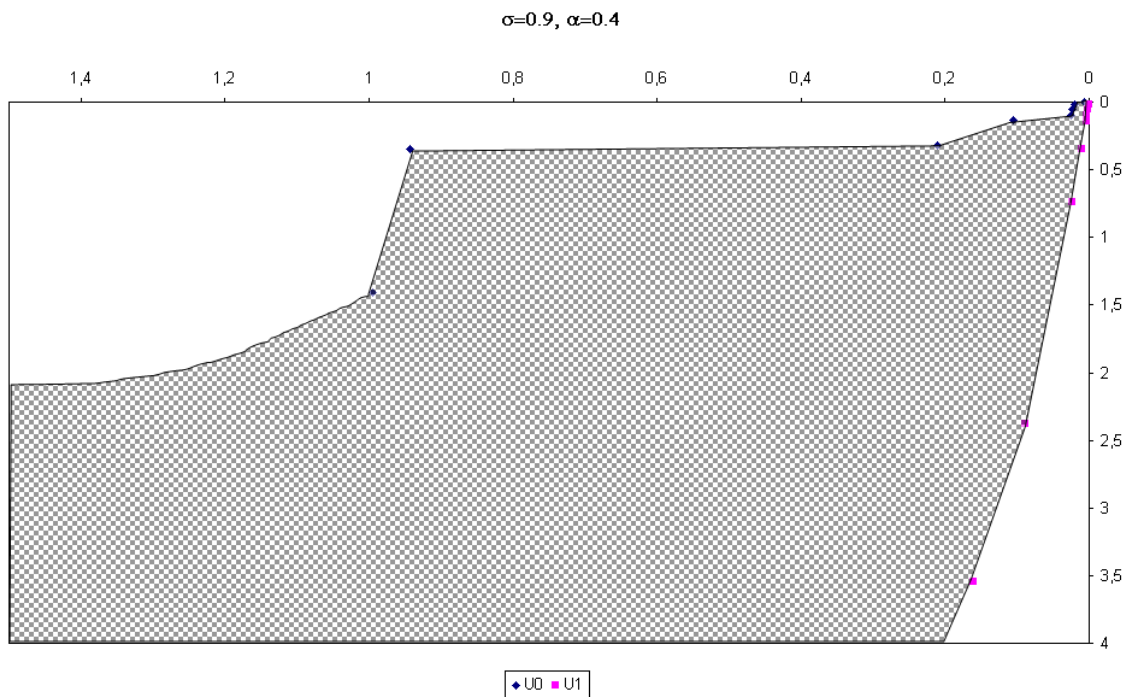


Figure 7: set of individual endowments such that default Pareto improves ex-ante welfare when $\alpha = 0.2$ and $\sigma = 0.9$.

In figure 7 the set of endowments for which default Pareto improves welfare is the squared-grey shaded area, whose size is even larger than the green area in figure 5.2. Note that in this case the surface of the rose surface occupies more than half of the surface of plane $V_1$.

Interestingly, one interpretation of the last numerical results would be that introducing defaultable securities (when the default occurs without stigma ex-post in the bad states of the world) is more *likely* to be Pareto improving in economies where the likelihood of default is higher and with more risk averse agents (assuming symmetric preferences). Being less rigorous with the interpretation, such results suggest that more "fragile" economies (in the sense of being

more prone to adverse shocks in future output and being populated with more risk-averse investors) should be more prone to develop defaultable securities than less "fragile" economies, as long as this default does not arise from factors ignored in this paper such as asymmetric information or credibility issues, factors that may induce the arising of a very different type of default (more linked to a "fraud" type of default). In other words, the results in this section indicates that financial regulators in such "fragile" economies, in their efforts of protecting lenders, and also if guided by ex-ante Pareto-efficiency criterion, do not have enough justification to ban the trading of defaultable securities, but only justification of defaults that arise due to "strategic" decisions by debtors. Of course, this paper does not intend to provide a definite answer on financial regulation concerning defaultable securities. However, it does give a warning concerning possible policy proposals that ban the trading of such securities, without seeing the possible ex-ante benefits even for lenders facing no risk sharing needs.

# 6 Equilibrium portfolios: numerical examples

As stated in the introduction, The last exercise performed in this paper is the explicit computation of equilibria assuming logarithmic preferences and certain values for the probabilities. The main result from these computations is that, when the defaultable bond is allowed to be traded, then the equilibrium portfolio resembles that of some hedge funds, such as the Long Term Capital Management,[12]. This means purchasing (on behalf of investors) units of a defaultable security and short selling units of the riskless bond. Clearly this depends on the values of certain parameters, such as the agents' beliefs, but at least it shows that such portfolio decisions are not necessarily inconvenient even from the perspective of investors not facing any needs of hedging risks.

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[12]For example, Edwards (1999) states: "*LTCM was primarily engaged in what hedge fund practitioners call 'market neutral arbirtage'. Its main holdings appear to have been long positions that it considered undervalued and short positions in bonds that it considered overvalued. More specifically, it bought (...)* **high-yielding**, *less liquid bonds, such as Danish mortgage-backed securities, bonds issued by emerging market countries, and* **'junk' corporate bonds**, *and sold short (...) low-yielding, more liquid bonds, such as* **U.S. government bonds**." (Bold letters are ours).

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Throughout this section we assume the following functional form for the utility function:

$$u(x) = \ln x$$

## 6.1    Example 1

This example considers a value for $\alpha = 0.2$ and each agent's endowments:

| agent-state | 0 | 1 | 2 |
|---|---|---|---|
| **1** | 1.5 | 0.25 | 0.25 |
| **0** | 0.5 | 0.1 | 4 |

Assuming that agents trade in both securities the equilibrium prices and portfolio decisions are as follows:

| $q_0^*$ | $q_1^*$ | $b_0^{0*}$ | $b_0^{1*}$ | $b_1^{0*}$ | $b_1^{1*}$ |
|---|---|---|---|---|---|
| 0.376 | 1.519 | -1.833 | 0.094 | 1.833 | -0.094 |

This clearly shows that the agent 1 (the "lender") purchases about 1.83 of the defaultable security and short sell 0.09 units of the riskless asset.

## 6.2    Example 2

The last example corresponds to a configuration of endowments that are not obviously close to the set of default-free equilibrium allocations. Thus we cannot ensure that the latter allocation, though Pareto-efficient, implies a Pareto improvement from any default free equilibrium. The next example presents a different pattern for endowments that is close enough to one default-free equilibrium allocation and still generates (when introducing the defaultable security) a portfolio that implies purchasing the risky asset and selling short the riskless one. Assume still that $\alpha = 0.2$ but now endowments are:

| agent-state | 0 | 1 | 2 |
|---|---|---|---|
| **0** | 2.627 | 0.1 | 4 |
| **1** | 1.5 | 0.25 | 0.25 |

These endowments constitute an equilibrium allocation when the riskless asset is the only one traded. When adding the defaultable security the equilibrium becomes:

| $q_0^*$ | $q_1^*$ |
|---|---|
| 0.776898396 | 3.135339954 |

| $b_0^{0*}$ | $b_0^{1*}$ | $b_1^{0*}$ | $b_1^{1*}$ |
|---|---|---|---|
| -1.079036575 | 0.161398678 | 1.079036575 | -0.161398678 |

Thus, in this equilibrium agent 1 (the lender) purchases about 1.08 units of the defaultable bond at the price of 0.776, while selling short 0.16 units of the riskless security at a price of 3.135. Clearly, the price of every riskless security must be larger than that of the risky security (both denominated in units of the consumption good) to generate such incentives by agents.

## 6.3 Example 3

Examples 1 and 2 assume a value for $\alpha = 0.2$,where the difference remained on the endowments side. This third example keeps the same endowment patterns as in example 2 (and so they are a default-free equilibrium allocation) but increasing the value of $\alpha$ to 0.4.This means that the default state (state 1) is more likely in this example than in the last one. In this case the equilibrium is as follows

| $q_0^*$ | $q_1^*$ |
|---|---|
| 0.282352941 | 1.349019608 |

| $b_0^{0*}$ | $b_0^{1*}$ | $b_1^{0*}$ | $b_1^{1*}$ |
|---|---|---|---|
| -1.607598039 | -0.08125 | 1.607598039 | 0.08125 |

Note now that agent 1 purchases *both* securities. Note the drop of the equilibrium price in both securities in this example compared to example 2. This is essentially what seems to be driving the reversion in the sign of $b_1^{1*}$ relative to the last example. It is clear then that the pattern of purchasing the risky asset and selling short the riskless one cannot be considered as universally valid.

# 7 Conclusion

This paper emphasizes that, as it is, an extremely basic version of the standard general equilibrium analysis with incomplete markets developed after Arrow (1953 and 1964) can answer the question of when default improves ex-ante welfare even for lenders who do not face intrinsic risk sharing needs. The main proposition of this paper states that when endowments are not too far from those allocations satisfying the equalization across lender's intertemporal marginal rate of substitutions and borrower's expected intertemporal marginal rate of substitutions (when borrowers do face uncertainty in their future income) then a default that imposes no punishment when effectively executed (in the state with low income for the borrower) implies a Pareto improvement. In fact, numerical computations show that, generically, the set of economies where this default property holds is not negligible and that it is larger the more risk averse and the higher is the probability of default. This conclusion seems to have been largely ignored in the macroeconomics literature, especially in the light of important policy discussions regarding regulation of trading on derivative and other related financial markets.

It is important to emphasize that this paper uses a *non-strategic* concept of default. Therefore, the interpretation given above should be taken with care. The main result of this paper does not necessarily mean that any type of default (in particular, strategic default) may imply a Pareto improvement. It only states that non-strategic default (i.e., default as a consequence of lack of resources to repay debt) may imply an improvement in ex-ante welfare for all agents. When considering strategic default (i.e., fraud, moral hazard, etc.) then this conclusion may not hold anymore. This consideration matters especially when looking at the empirical literature linking investor protection and the quality of institutions (see, e.g., La Porta et al., 1998).

As a by-product, special cases of the model predict that in such default equilibrium lenders end up purchasing the defaultable security and short sell units of the default free asset. Thus, if an investment fund must decide a portfolio of those two assets on behalf of such lenders, they end up deciding portfolios very similar to those of some famous hedge funds such as LTCM. Thus, the policy of "purchasing the risky asset, selling short the riskless one" does not imply an inefficiency per-se, at least in qualitative terms. Of course, information and bounded rationality considerations may introduce distortions in the form of "purchasing too much of the risky security" and/or "selling too much of the riskless asset". However this is not a qualitative but

a quantitative issue. Thus it is important to differentiate between these two questions when discussing regulations in the hedge and mutual fund industry.

Clearly, this model introduces an important number of simplifying assumptions, such as a unique physical commodity and only two types of agents. For example, as stated in the introduction, Hart (1975) found with more than one commodity that the introduction of a new asset may imply a Pareto worsening rather than an improvement, due to relative price effects. Therefore, it is important to see whether the same type of argument can be applied to a world with many goods. Similarly, a higher degree of heterogeneity may imply a more subtle characterization of endowments where the Pareto improvement result holds. The extension to more general models involving an arbitrary number of goods and consumers should clearly be the subject of further research.

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